CHAOS MEASURE IN AUTONOMOUS LPA MODEL

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Abstract

The discrete autonomous LPA model of dynamical systems investigated for regular and chaotic evolutions under different feasible conditions in the framework of nonlinear dynamics. Evolutionary phenomena discussed through bifurcation analysis leading to chaos. As part of chaos measure, numerical calculations performed to obtain Lyapunov characteristic exponents (LCE), Topological Entropy, correlation dimension etc. The results obtained by numerical calculations are demonstrated through various graphics. Chaotic evolutions discussed at critical set of parameters, which reveals very significant results. Correlation dimension, which provides dimensionality of an attractor (Strange/Chaotic), obtained numerically by the use of certain statistical method.

Keywords: Autonomous LPA model, Bifurcation, Lyapunov Characteristic Exponents, Topological Entropy, Correlation Dimension.

1 Introduction

Mathematical models expressing real phenomena are mostly nonlinear in nature. Their evolutionary dynamical behavior often shows properties like unpredictability and chaos attracting researchers obtaining interesting results [4; 5; 24]. The model on population dynamics and ecology are frequently used models and most considerable problems in dynamical systems. Investigators generally prefer to use difference equation while describing mathematical models in context of biological models. Numerous articles have appeared on such models after publication of articles by R. May [17; 18] with reasonable assumption of evolution processes of population in concerned. Such studies generate quite interesting results.

Many nonlinear systems exhibit chaos in some parameter space but in some cases within the system because of the interaction among different agents, complexity character also visible during evolution. Unpredictable motion is thus a mix phenomenon of chaos and complexity. Presence of complexity is responsible of coexistence of multiple attractors, bistability, intermittency, cascading effects, exhibit of hysteresis properties, and some more properties, [3; 6; 9; 26]. Chaotic evolution measured by positivity of Lyapunov exponents, (LCEs), whereas its negative value signifies the system is regular, [2; 8; 13]. Complexity measured by increase of topological entropy; more increase in topological entropy signifies the system is more complex, [7; 14; 15; 21; 22; 27].

Evolution in insects considered metamorphosis since big changes observed during their growth and development. Insect evolution passes through four clearly different stages: egg, larvae, pupae and adult. Class of such insects listed as butterflies, moths, beetles, flies, bees, wasps, and ants. Changing from one stage to another an insect has to molt its skin and each time it emerge larger and of different form until it reaches the adult stage, [10]. Some recent articles dedicated for studying particular LPA model for insect's fluctuations, like flour beetle Tribolium, with reasonable approaches draw very interesting results, [1; 11; 12; 25].

In the present work we have considered autonomous LPA model discussed in Henson et al, [25]. The objective here to study further the chaotic behavior and presence of complexity in insect population under different conditions and with detailed numerical approach. Bifurcation phenomena observed here by varying certain parameter while keeping fixed value for other parameter. Numerical investigations further extended to calculate attractors, Lyapunov exponents, topological entropies and correlation dimension of attractors. We conclude with discussion by analyzing the results obtained through this investigation.

2 The Autonomous LPA Model

The autonomous LPA model used here defined by the set of three equations, Henson et al., [25]:

$$\begin{cases} L_{t+1} &= b\, A_t\, e^{(-c_{ea}\, A_t - c_{el}\, L_t)}, \\ P_{t+1} &= (1-\mu_l)\, L_t, \\ A_{t+1} &= P_t\, e^{(-c_{pa}\, A_t)} + (1-\mu_a)\, A_t, \end{cases} \tag{1}$$

where L_t, P_t, A_t denote the population variable representing, respectively, the number of Larvae, Pupae and Adult flour beetles at discrete time t. The discrete time interval represents the time taken for a larva to mature to pupae. Parameter $b > 0$ is the average number of larvae (eggs), recruited per adult per unit time in the absence of cannibalism, $0 < \mu_a < 1$ and $0 < \mu_l < 1$ are the probabilities of deaths, except cannibalism, for an adult and larvae respectively. The exponential factors represent probabilities of survival of individual's cannibalism per unit time and c_{el}, c_{ea}, c_{pa} all positive called cannibalism coefficients.

3 Bifurcation Analysis

Bifurcations in a system are changes occurring in the qualitative structure of the system during evolution when a particular parameter of it varies while other parameters kept constant. One observes sudden change during process of changing values of parameters in some specified way. Bifurcation scenario clearly displays regular and chaotic evolution of the system, [19; 20]. Qualitative structures of different parameter space are different and so the bifurcation diagrams.

Bifurcation diagram, of Fig. 1, obtained by varying $0 < \mu_a < 1$ and fixing values of parameters $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_l = 0.2$, $c_{pa} = 0.005$, with initial conditions $L_0 = 0.1$, $P_0 = 0.1$, $A_0 = 0.1$. We observe stable limit cycle as $\mu_a > 0.2$ and then two cycles after $\mu_a = 0.22$. Then bifurcation diagram shows chaos which continues till $\mu_a = 0.45$ and then onwards again regular behaviour seen at $\mu_a = 0.49$.

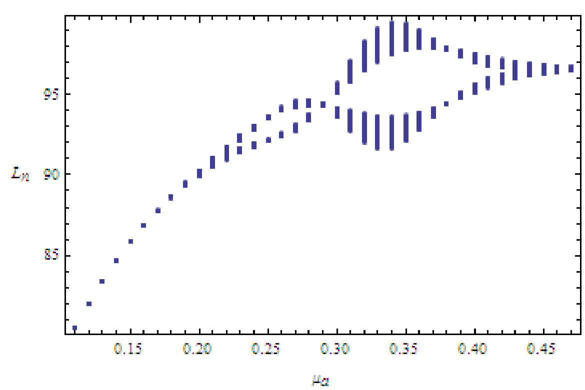

Figure 1: Bifurcation diagram of LPA model for the parameter values $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_l = 0.2$, $c_{pa} = 0.005$, and $0 < \mu_a < 1$.

In Fig. 2, bifurcation diagrams plotted on different axes of larvae, pupae and adults with the same parameters used in Fig. 1.

Simulation results of LPA model with initial conditions $L_0 = 0.1$, $P_0 = 0.1$, $A_0 = 0.1$ and keeping $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $c_{pa} = 0.005$, constant obtained and presented in tabular form in Table- 1.

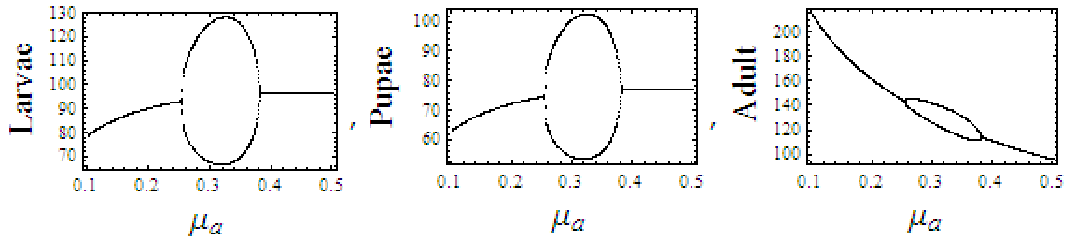

Figure 2: Bifurcation diagram of (a) Larvae (b) Pupae (c) Adult flour Beetles.

Table 1: Dynamical behaviour

Parameter Kept Constant	Parameter varied	Range of varied parameter	Dynamical behaviour
$\mu_l = 0.2$	μ_a	0- 0.2	Stable limit cycle
$b, c_{ea}, c_{el}, c_{pa}$	$0 < \mu_a < 1$	0.21-0.25	Two cycle
		0.26-0.45	Chaos
		0.46-0.49	Stable limit cycle
		0.5-1.0	Extinction
$\mu_a = 0.96$	μ_l	0-0.5	Stable limit cycle
$b, c_{ea}, c_{el}, c_{pa}$	$0 < \mu_l < 1$	0.51-0.8	Chaos
		0.8-1.0	Stable limit cycle

Again for $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.96$, $c_{pa} = 0.005$ and $0 < \mu_l < 0.5$, an strange type of bifurcation observed as shown in Fig. 3.

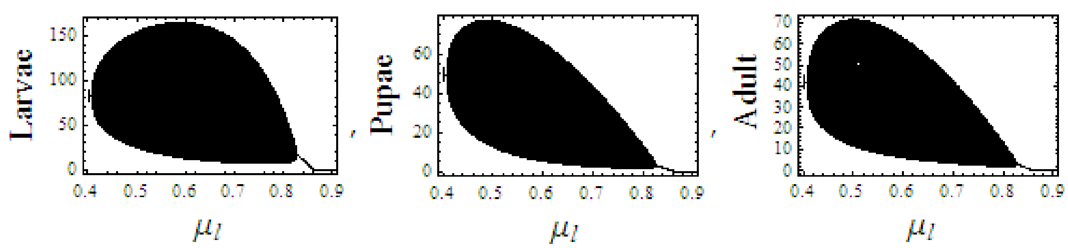

Figure 3: Bifurcation diagram of (a) Larvae (b) Pupae (c) Adult flour Beetles with the parameter values $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.2$, $c_{pa} = 0.005$, and $0 < \mu_l < 1$.

4 Regular and Chaotic Attractors

In this section we have plotted the attractors for the LPA Model with different parameters to study the regular and chaotic behavior of the system. While the regular attractor we observe a limit cycle and the pattern is very simple to analyze, a chaotic attractor is composed of a complex pattern. In fig. 4, we have plotted the chaotic attractor using the parameters $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.26$, $\mu_l = 0.2$ and $c_{pa} = 0.005$.

Evolution of system (1) occur through the phenomena of Hopf bifurcation. This one can observe analyzing the following figures, Fig. 5 and Fig. 6, where a stable fixed point evolve into a stable limit cycle due to change of parameter.

For a certain set of parameter the system evolve chaotically. A chaotic time series attractor is shown below in Fig. 5.

5 Lyapunov Characteristic Exponent

Regular and chaotic evolutions are perfectly identified by measuring Lyapunov characteristic exponents (LCEs). Evolution, where LCE < 0, called regular and where LCE > 0, called chaotic. Actually, whenever two orbits initiated with infinitesimal separation, the LCE provides us a standard measure of exponential divergence.

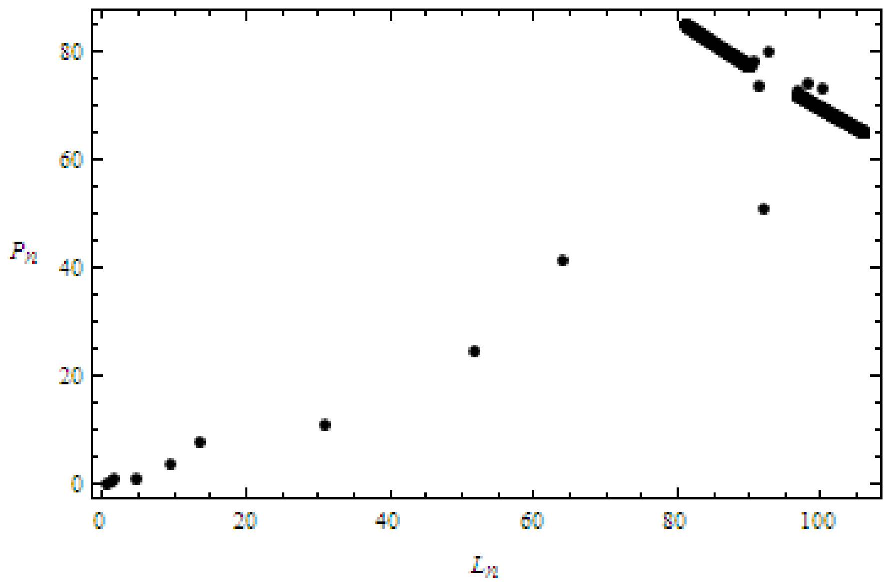

Figure 4: Attractor of LPA model for the parameter values $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.26$, $\mu_l = 0.2$ and $c_{pa} = 0.005$.

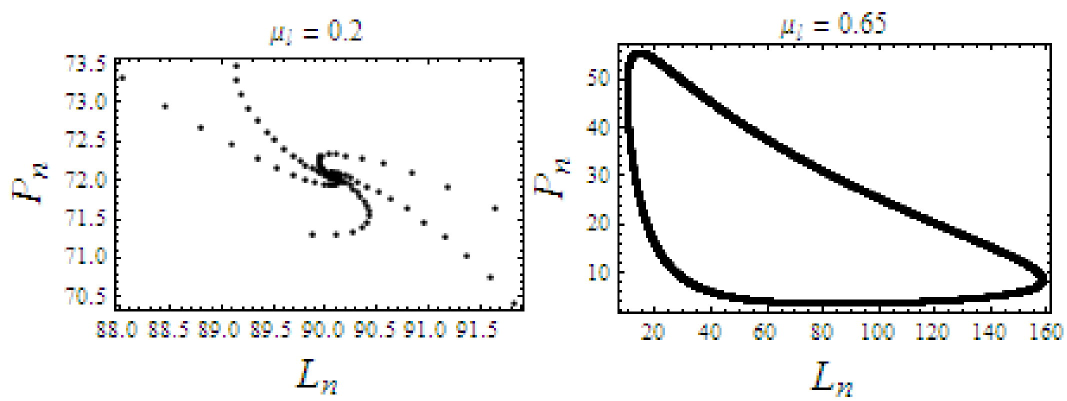

Figure 5: Attractor of LPA model for the parameter values $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.96$, $\mu_l = 0.2$ and $c_{pa} = 0.005$.

For one dimensional system, if $\delta\,x_0$ denotes the initial separation and, if $\delta\,x(t)$, denotes separation at time t, then, we have

$$|\delta\,x(t)| \approx e^{\lambda\,t}\,|\delta\,x_0| \tag{2}$$

Here, λ called the Lyapunov exponent. Similar rule exists for higher dimensional system. For numerical calculations of LCEs, a systematic way described in recent literature, [16; 23]. For any system which is described by a map, is regular for $\lambda \leq 0$ and when $\lambda > 0$, the system is chaotic. As shown in Fig. 7, we have plotted LCEs for two cases. Figure (a) show initially chaotic evolution which changes to regular with increasing iterations. Figure (b) is of interesting type. This plot depicts the bifurcation case shown in Fig. 3.

6 Topological Entropy

The measure of complexity is provided by the Topological entropy. More increase in topological entropy signifies the system is more complex, [3; 6; 9; 26]. The topological entropy discussed here closely related to that of Li and Yorke chaos and a theoretical development given in [15]. For the numerical calculations of topological entropy, we follow the setup of the recent article [22]. More complexity of a system signifies that the system is having more

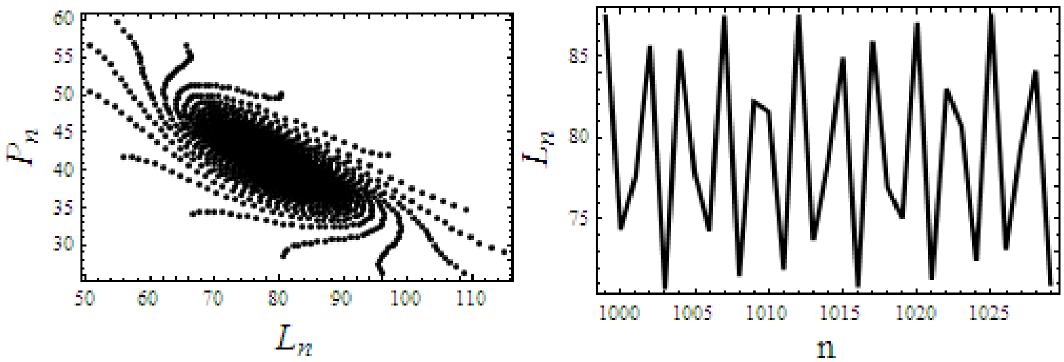

Figure 6: For $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.96$, $\mu_l = 0.48$ and $c_{pa} = 0.005$ chaotic time series is formed.

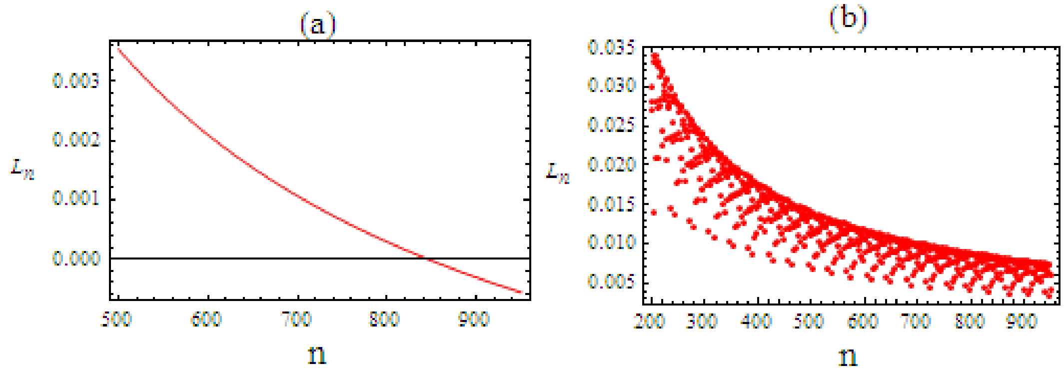

Figure 7: LCE diagrams 0f LPA model (a) left Figure, with parameter values $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.4$, $\mu_l = 0.2$ and $c_{pa} = 0.005$ and (b) right figure with parameter values $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.96$, $\mu_l = 0.5$ and $c_{pa} = 0.005$.

topological entropy. For system (1), topological entropies have been obtained for different values of μ_a and μ_l, and shown in Fig. 8. From Topological Entropy plot, we can observe the increase in topological entropy in the zones of complexity when the parameter value of $0.2 < \mu_a < 0.4$ and $0.8 < \mu_l < 0.9$.

One observes from these plots that the complexity exist in case (a) in $0 < \mu_a < 0.45$ and in case (b) in a very small interval $0.855 < \mu_l < 0.865$.

7 Calculation of Correlation Dimension

To measure the dimensionality of a system of the space occupied by a set of random points, we use a statistical measure called correlation dimension, [13]. If for a deterministic dynamical system, we wish to detect any indication of the existence of chaotic attractor, we check for a fractional value of this dimension. To calculate correlation dimension we use an accepted method discussed in recent articles, [15; 16; 22; 23]. For a map $f : U \to U$, if we choose an orbit $O(x_1) = \{x_1, x_2, x_3, x_4, - - - -\}$, the correlation dimension of this orbit $O(x_1)$ can be computed for a given positive real number r, by forming the correlation integral, [15; 16],

$$C(r) \quad = \quad \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{n} H\left(r- \parallel x_i - x_j \parallel\right), \tag{3}$$

where

$$H(x) \quad = \quad \begin{cases} 0, & x < 0, \\ 1, & x \geqslant 0, \end{cases}$$

is the unit-step function, (Heaviside function), U is an open bounded set in R^n, the number of pairs of vectors which are closer to r when $1 \leq i, j \leq n$ and $i \neq j$, is indicated by the summation in the equation and the density of pair of clearly different vectors x_i and x_j that are closer to r is measured by $C(r)$. Once we calculate $C(r)$, the correlation dimension D_c of $O(x_1)$ which is defined as

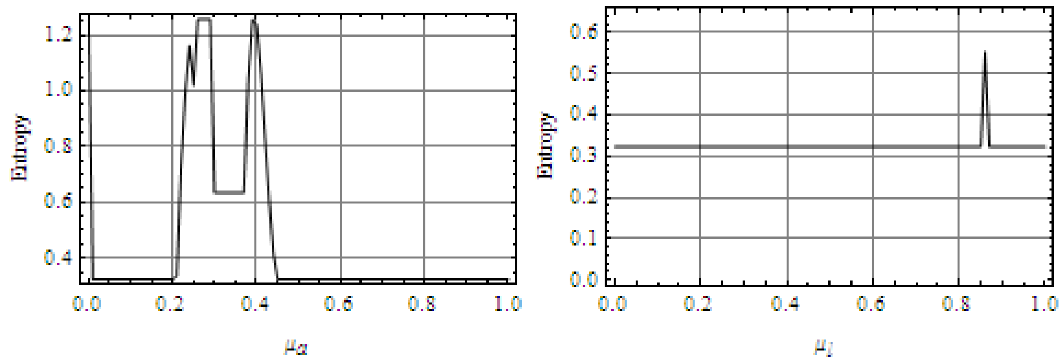

Figure 8: Plots of topological entropies for parameter values in (a) $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $0 \leq \mu_a \leq 1$, $\mu_l = 0.2$ and $c_{pa} = 0.005$ and in (b) $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.96$, $0 \leq \mu_l \leq 1$ and $c_{pa} = 0.005$.

$$D_c \quad = \quad \lim_{r \to \mathbf{0}} \frac{log\, C(r)}{log\, r}, \tag{4}$$

will be obtained, after plotting $log\, C(r)$ against $log\, r$ and then by fitting a straight line to this curve. From the equation of the straight line fitted to the correlation curve, the $y-$intercept will provide the value of the correlation dimension D_c.

We have calculated the correlation integrals $C(r)$ for models (1), and demonstrated, as correlation curves, through graphics Fig. 9 below by changing the parameter $\mu_a = 0.26$, $\mu_l = 0.2$ and $\mu_a = 0.96$, $\mu_l = 0.65$.

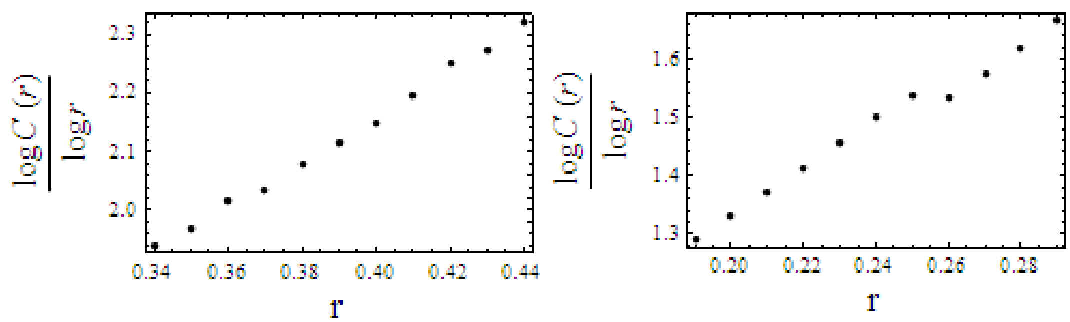

Figure 9: Correlation curves of LPA Model by using the parameter values (a) $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.26$, $\mu_l = 0.2$ and $c_{pa} = 0.005$ and (b) $b = 6.9$, $c_{ea} = 0.01$, $c_{el} = 0.01$, $\mu_a = 0.96$, $\mu_l = 0.65$ and $c_{pa} = 0.005$.

By using least square linear fit method, we have obtained the equations of straight lines appropriately fitting these correlation curves. For the curve shown in Fig. 9, the equation of the straight line obtained as
(a) y = 0.628946 + 3.82901 x, and (b) y = 0.616733 + 3.60358 x.

The $y-$intercept of these are, respectively, (a) $0.628946 \approx 0.63$, (b) $0.616733 \approx 0.62$. Thus the correlation dimensions are, respectively, (a) $D_c \cong 0.63$, and (b) $D_c \cong 0.62$.

8 Conclusion

Our objective of investigation of LPA system (1) is to study chaos and complexity behavior during its evolution. Bifurcation diagrams shown in figures, Fig. 1 – Fig. 3, indicate some interesting evolution scenario when certain parameter allowed varying while keeping others fixed. Plots shown in Fig. 5 and Fig. 6, indicate the system evolve through Hopf bifurcation. LCEs plot in Fig. 7(b) reflects the case of bifurcation of Fig. 3. Complexity exists in the system in case (a) in $0 < \mu_a < 0.45$ but very little in case (b) only in a small interval $0.855 < \mu_l < 0.865$. It has been observed that complexity may exist even if the system is regular and also, it may or may not exist for chaotic cases. Correlation dimensions $D_c \cong 0.63$ and $D_c \cong 0.62$ calculated for two different cases of parameter spaces, as shown in Fig. 9. For a general class of insects model studied here could be modified and significant analysis would be incorporated in future investigations.

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