Numerical Solutions of Fuzzy Differential Equation under Generalized Differentiability concept using a Third Order Runge - Kutta Method with Contra Harmonic Mean

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Abstract:

This paper presents a solution for first-order Fuzzy Differential Equation (FDE) under generalized differentiability concept by Runge-Kutta Third order with Contra Harmonic Mean (CHM) in which, new parameters are taken in k_2 and k_3 to increase the order of accuracy of the solution. The precision and efficiency of the proposed method is portrayed by resolving a first-order FDE.
Keywords: *Fuzzy Differential Equation, Numerical solution, Runge-Kutta Third order, Contra Harmonic Mean.*

1. Introduction

Fuzzy Differential Equation (FDE) models have a extensive range of applications in different disciplines of engineering and in the medicine domain. The numerical solution of FDEs plays a vital role in science and engineering. In recent times, many research papers are focused on numerical solutions of fuzzy initial value problems (FIVPS). The application of the Euler method for finding the numerical solution of FIVPS was studied in [1].A variety of numerical methods have been developed for finding the solution of FIVPs, and some of them can be found in [2-6].But initial value problems studied through the concept Hukuhara differentiability have many solutions that have an increasing length of support as the independent variable increases. Bede & Gal [7] introduced a new concept of fuzzy derivatives called the generalized differentiability of fuzzy interval-valued functions. In this setting, the solution of an FDE may have a decreasing length of support as the independent variable increases. However, it depends on the selection of the fuzzy derivatives and hence the uniqueness is not ensured. Bede [7] proved a characterization theorem which states that under certain conditions, FDEs is equivalent to a system of ordinary differential equations. This generalisation was further improved by Chalco-Cano & Roman-Flores [8]. The numerical solution of FIVPS under generalized concept can be found in [9-22].

This paper is organized as follows. In section 2, we provide some background on fuzzy numbers and FDEs. In section 3, we develop RK-third order Contra Harmonic Mean with new parameters for ordinary differential equations. In section 4, the proposed method is extended to FDEs. In section 5, we present a numerical example, and the conclusion is in section 6.

2. Preliminaries

Definition 2.1[7] A fuzzy number u is a fuzzy subset of R (i.e) $u: R \to [0,1]$satisfying the following conditions:

1. u is normal (i.e.) $\exists x_0 \in R$ with $u(x_0) = 1$,
2. $u$is a convex fuzzy set (i.e.)$u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\} \forall \lambda \in [0,1],\ x, y \in R$,
3. $u$is upper semi continuous on R.
4. $\{x \in R, u(x) > 0\}$is compact.

Let E be the class of all fuzzy subsets of R. Then E is called the space of fuzzy numbers. Clearly,$R \subset E$ and $R \subset E$ is understood as$R = \{\mathcal{R}_x : \mathcal{R}\text{is usual real number}\}$.

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$ that satisfies the following requirements

1. $\underline{u}(r)$is a bounded left continuous non decreasing function over [0, 1],with respect to any r.
2. $\overline{u}(r)$ is a bounded right continuous non increasing function over [0, 1],with respect to any r.
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

Then the r-level set is$[u]_r = \{x \backslash u(x) \geq r\}$, $0 < r \leq 1$ is a closed and bounded interval denoted by $[u]_r = [\underline{u}(r), \overline{u}(r)]$. And clearly,$[u]_0 = \{x \backslash u(x) > 0\}$ is compact.

Let $P_K(R)$ denote the family of all nonempty compact convex subsets of R. Then the r-level sets $[u]_r$ are in $P_K(R)$ for $0 \leq r \leq 1$.

Let $d_H(A, B)$ be the Hausdorff distance between sets$A, B \in P_K(R)$.
Then $d(u, v) = \sup_{0 \leq r \leq 1}([u]_r, [v]_r)$is a metric in E and (E, d) is a complete metric space .

For$x, y \in E$ if there exists an$z \in E$ such that $x = y + z$, then z is called the H-difference of $x$and y and is denoted by $x - y$.

Definition 2.2 [7] Let $F: (a, b) \to E$ and $x_0 \in (a, b)$.It is said that F is strongly generalized differentiable on x_0, if there exists an element $F'(x_0) \in E$, such that

(i) for all $h > 0$ sufficiently small,$\exists F(x_0 + h) - F(x_0)$,$F(x_0) - F(x_0 - h)$ and the limits (in the metric d) $\lim_{h \searrow 0} \frac{F(x_0+h)-F(x_0)}{h} = \lim_{h \searrow 0} \frac{F(x_0)-F(x_0-h)}{h} = F'(x_0)$,

(or)

(ii) for all $h > 0$ sufficiently small,$\exists F(x_0) - F(x_0 + h)$, $F(x_0 - h) - F(x_0)$ and the limits $\lim_{h \searrow 0} \frac{F(x_0)-F(x_0+h)}{(-h)} = \lim_{h \searrow 0} \frac{F(x_0-h)-F(x_0)}{(-h)} = F'(x_0)$,

(or)

(iii) for all $h > 0$ sufficiently small, $\exists F(x_0 + h) - F(x_0)$, $F(x_0 - h) - F(x_0)$ and the limits $\lim_{h\searrow 0} \frac{F(x_0+h)-F(x_0)}{h} = \lim_{h\searrow 0} \frac{F(x_0-h)-F(x_0)}{(-h)} = F'(x_0)$,

(or)

(iv) for all $h > 0$ sufficiently small, $\exists F(x_0) - F(x_0 + h)$, $F(x_0) - F(x_0 - h)$ and the limits $\lim_{h\searrow 0} \frac{F(x_0)-F(x_0+h)}{(-h)} = \lim_{h\searrow 0} \frac{F(x_0)-F(x_0-h)}{h} = F'(x_0)$.

($h$and $(-h)$at denominators mean $\frac{1}{h}$ and $-\frac{1}{h}$, respectively).

Remark 2.1 For our simplicity, we say that the fuzzy- valued function F is $(i)-$differentiable if it satisfies case (i) of Definition 2.2 and $(ii)-$differentiable if it satisfies case (ii) of Definition 2.2

Following the ideas of Bede & Gal [7], Chalco-Cano & Roman-Flores [8] defined the fuzzy lateral H-derivative for a fuzzy mapping $F: (a, b) \to E$ as follows:

Definition 2.3[8] Let be $F: (a, b) \to E$ and $x_0 \in (a, b)$ .We say that,

(i) if there exists an element $F'(x_0) \in E$ such that,for all $h > 0$ sufficiently near to 0, there are $F(x_0 + h) - F(x_0)$, $F(x_0) - F(x_0 - h)$ and the limits (in the metric d) $\lim_{h\to 0^+} \frac{F(x_0+h)-F(x_0)}{h} = \lim_{h\to 0^+} \frac{F(x_0)-F(x_0-h)}{h} = F'(x_0)$.

(or)

(ii) if there exists an element $F'(x_0) \in E$ such that,for all $h < 0$ sufficiently near to 0, there are $F(x_0 + h) - F(x_0)$, $F(x_0) - F(x_0 - h)$ and the limits (in the metric d) $\lim_{h\to 0^-} \frac{F(x_0+h)-F(x_0)}{h} = \lim_{h\to 0^-} \frac{F(x_0)-F(x_0-h)}{h} = F'(x_0)$.

Theorem 2.1[8] Let $F: I \to E$ be a function and denote $[F(t)]_r = [f_r(t), g_r(t)]$,for each $r \in [0,1]$. Then

(i) if $F$is differentiable in the first form (i) of Definition 2.3, $f_r(t)$ and $g_r(t)$ are differentiable functions and $[F'(t)]_r = [f'_r(t), g'_r(t)]$.

(ii) If $F$is differentiable in the second form (ii) definition 2.3, $f_r(t)$ and $g_r(t)$ are differentiable functions and $[F'(t)]_r = [g'_r(t), f'_r(t)]$.

3. Third Order Runge-Kutta Formula with Contra Harmonic Mean

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)); t_0 \le t \le b \\ \quad y(t_0) = y_0 \end{cases} \tag{3.1}$$

The basis of all Runge-Kutta methods is to express the difference between the values of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=0}^{m} w_i k_i \tag{3.2}$$

where w_is are constants for all i

and $$k_i = hf(t_n + a_i h, y_n + \sum_{j=1}^{i-1} c_{ij} k_j) \tag{3.3}$$

with$h = t_{n+1} - t_n$ and $a_1 = c_{11} = 0$.

Most efforts to increase the order of accuracy of the Runge-Kutta methods have been accomplished by increasing the number of Taylor's series terms used and thus the number of functional evaluations required. The process proposed by Goeken and Olin Johnson [26] introduces new terms involving higher-order derivatives of 'f' in the Runge-Kutta k_i terms $(i > 1)$ to obtain a higher order of accuracy without a corresponding increase in evaluations of 'f', but with the addition of assessment of f'.

3.1 Third order Runge-Kutta Method with Contra Harmonic Mean

Consider the Runge-Kutta method with two slopes

$$y(t_{n+1}) = y(t_n) + w_1\left[\frac{k_1^2+k_2^2}{k_1+k_2}\right] + w_2\left[\frac{k_2^2+k_3^2}{k_2+k_3}\right] \quad (3.4)$$

Where $k_1 = hf(t_n, y(t_n))$ (3.5)

$$k_2 = hf\{t_n + c_{21}h + c_{22}h^2 f_y(t_n, y(t_n)), y(t_n) + a_{21}k_1 + ha_{22}[f_y(t_n, y(t_n))k_1\} \quad (3.6)$$

$$k_3 = hf\{t_n + c_{31}h + c_{32}h^2 f_y(t_n, y(t_n)), y(t_n) + a_{31}k_1 + a_{32}k_2 + ha_{33}f_y(t_n, y(t_n))k_1 + ha_{34}k_2 f_y(t_n, y(t_n))\} \quad (3.7)$$

The parameters $w_1, w_2, c_{21}, c_{22} a_{21,} a_{22}, c_{31}, c_{32}, a_{31}, a_{32}, a_{33}, a_{34}$ chosen to make y_{n+1} closer to $y(t_{n+1})$. There are 12 parameters to be determined. The Taylor's series expansion of Equation (3.4) aboutt_n gives:

Solving the above equations using Mathematica(write the version),we get

$$w_1 = \frac{1}{5}, w_2 = \frac{4}{5}, c_{21} = \frac{1}{3}, c_{22} = \frac{-1}{90}, a_{21} = \frac{1}{3}, a_{22} = \frac{-1}{90}, c_{31} = \frac{5}{6}, c_{32} = \frac{1}{18}, = \frac{-7}{24},$$
$$a_{32} = \frac{9}{8}, a_{33} = \frac{-53}{180}, a_{34} = \frac{7}{20}$$

Therefore from Equations (3.4) to (3.7), Third order Runge-Kutta method with contra harmonic mean is obtained as

$$y(t_{n+1}) = y(t_n) + \frac{1}{5}\left[\frac{k_1^2+k_2^2}{k_1+k_2}\right] + \frac{4}{5}\left[\frac{k_2^2+k_3^2}{k_2+k_3}\right] \quad (3.8)$$

Where$k_1 = hf(t_n, y(t_n))$ (3.9)

$$k_2 = hf\left\{t_n + \frac{1}{3}h - \frac{1}{90}h^2 f_y(t_n, y(t_n)), y(t_n) + \frac{1}{3}k_1 - \frac{h}{90}[f_y(t_n, y(t_n))k_1\right\} (3.10)$$

$$k_3 = hf\left\{t_n + \frac{5}{6}h + \frac{1}{18}h^2 f_y(t_n, y(t_n)), y(t_n) - \frac{7}{24}k_1 + \frac{9}{8}k_2 + \frac{53h}{180}f_y(t_n, y(t_n))k_1 + \frac{7h}{20}k_2 f_y(t_n, y(t_n))\right\} \quad (3.11)$$

Here$hf' = f_y(t_n, y(t_n))k_1 + hf_t(t_n, y(t_n))$.

4. Third-order Runge-Kutta Method Based on Contra Harmonic Mean for FIVPs

In this section, we will present the Third order Runge-Kutta method with Geometric Mean for the fuzzy differential equations. To solve the fuzzy system of ordinary differential system in $[t_0, t_1], [t_1, t_2], \dots, [t_k, t_{k+1}], \dots$, for $r \in [0,1]$,we will replace each interval $[t_k, t_{k+1}]$by a set of $N_k + 1$ regularly spaced points. The grid points on$[t_k, t_{k+1}]$ will be $t_{k,n} = t_k + nh_k$ where $h_k = \frac{t_{k+1}-t_k}{N_k}$ and $0 \le n \le N_k$.Now we will give algorithms to numerically solve the system in $[t_0, t_1], [t_1, t_2], \dots, [t_k, t_{k+1}], \dots$

Algorithm: To approximate the solution of the fuzzy initial value problem given by the system of equations in Equation (3.1).

Case:1 (i)-Differentiability

Step: 1 Let $h_k = \frac{t_{k+1}-t_k}{N_k}$, $\underline{w_r}(t_{k,0}) = \alpha_0$ and $\overline{w_r}(t_{k,0}) = \alpha_1$

Step: 2 Let i=1.

Step:3 Let $\underline{w_r}(t_{k,i}) = \underline{w_r}(t_{k,i-1}) + \frac{1}{5}\left[\frac{\underline{k_1^2}+\underline{k_2^2}}{\underline{k_1}+\underline{k_2}}\right] + \frac{4}{5}\left[\frac{\underline{k_2^2}+\underline{k_3^2}}{\underline{k_2}+\underline{k_3}}\right]$ (4.1)

$$\overline{w_r}(t_{k,i}) = \overline{w_r}(t_{k,i-1}) + \frac{1}{5}\left[\frac{\overline{k_1^2}+\overline{k_2^2}}{\overline{k_1}+\overline{k_2}}\right] + \frac{4}{5}\left[\frac{\overline{k_2^2}+\overline{k_3^2}}{\overline{k_2}+\overline{k_3}}\right] \quad (4.2)$$

where $\underline{k_1} = \underline{f_r}(t_{k,i-1}, w_r(t_{k,i-1}))$; (4.3)

$$\overline{k_1} = \overline{f_r}(t_{k,i-1}, w_r(t_{k,i-1})); \quad (4.4)$$

$$\underline{k_2} = \underline{f_r}(t_{k,i-1} + \tfrac{1}{3}h + \tfrac{-1}{90}h^2 f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right), w_r(t_{k,i-1}) + \tfrac{1}{3}k_1 + \tfrac{-1}{90}h\left[f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right)k_1\right] \quad (4.5)$$

$$\overline{k_2} = \overline{f_r}(t_{k,i-1} + \tfrac{1}{3}h + \tfrac{-1}{90}h^2 f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right), w_r(t_{k,i-1}) + \tfrac{1}{3}k_1 + \tfrac{-1}{90}h\left[f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right)k_1\right] \quad (4.6)$$

$$\underline{k_3} = \underline{f_r}(t_{k,i-1} + \frac{5}{6}h + \frac{1}{18}h^2 f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right), w_r(t_{k,i-1}) + \frac{-7}{24}k_1 + \frac{9}{8}k_2 + \tfrac{-53}{180}h\left[f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right)k_1\right] + \tfrac{7}{20}h\left[f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right)k_2\right] \quad (4.7)$$

$$\overline{k_3} = \overline{f_r}(t_{k,i-1} + \frac{5}{6}h + \frac{1}{18}h^2 f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right), w_r(t_{k,i-1}) + \frac{-7}{24}k_1 + \frac{9}{8}k_2 + \tfrac{-53}{180}h\left[f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right)k_1\right] + \tfrac{7}{20}h\left[f_y\left(t_{k,i-1}, w_r(t_{k,i-1})\right)k_2\right] \quad (4.8)$$

Step: 4 $t_{k,i+1} = t_{k,0} + (i+1)h_k$

Step: 5 Let $i = i + 1$

Step: 6 If $i \leq N$, go to Step 3.

Step:7 The algorithm ends, and $[\underline{w_r}(t_{k+1}), \overline{w_r}(t_{k+1})]$ approximates the value of $[\underline{Y_r}(t_{k+1}), \overline{Y_r}(t_{k+1})]$.

Case:2 (ii)-Differentiability

Step: 1 Let $h_k = \frac{t_{k+1}-t_k}{N_k}$, $\underline{w_r}(t_{k,0}) = \alpha_0$ and $\overline{w_r}(t_{k,0}) = \alpha_1$

Step: 2 Let i=1.

Step:3 Let $\underline{w_r}(t_{k,i}) = \underline{w_r}(t_{k,i-1}) + \frac{1}{5}\left[\frac{\overline{k_1^2}+\overline{k_2^2}}{\overline{k_1}+\overline{k_2}}\right] + \frac{4}{5}\left[\frac{\overline{k_2^2}+\overline{k_3^2}}{\overline{k_2}+\overline{k_3}}\right]$; (4.9)

$$\overline{w_r}(t_{k,i}) = \overline{w_r}(t_{k,i-1}) + \frac{1}{5}\left[\frac{\underline{k_1^2}+\underline{k_2^2}}{\underline{k_1}+\underline{k_2}}\right] + \frac{4}{5}\left[\frac{\underline{k_2^2}+\underline{k_3^2}}{\underline{k_2}+\underline{k_3}}\right] \quad (4.10)$$

Where $\underline{k_1}$, $\overline{k_1}$, $\underline{k_2}$ $\overline{k_2}$, $\underline{k_3}$ and $\overline{k_3}$ are given as in equations(4.3) to (4.8).

Step: 4 $t_{k,i+1} = t_{k,0} + (i+1)h_k$

Step: 5 Let $i = i + 1$

Step: 6 If $i \leq N$, go to Step 3.

Step:7 The algorithm ends, and $[\underline{w_r}(t_{k+1}), \overline{w_r}(t_{k+1})]$ approximates the value of $[\underline{Y_r}(t_{k+1}), \overline{Y_r}(t_{k+1})]$.

Lemma 4.1 Let the sequence of numbers $\{W_n\}_{n=0}^{N}$satisfy$|W_{n+1}| \leq A|W_n| + B, 0 \leq n \leq N - 1$,or some given positive constants A and B, then$|W_n| \leq A^n|W_0| + B\frac{A^n-1}{A-1}, 0 \leq n \leq N-1$.
The proof of Lemma (4.1) follows Lemma 1 of Ming Ma et al [1].
Lemma 4.2 Let the sequence of numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \leq |W_n| + Amax\{|W_n|, |V_n|\} + B$, $|V_{n+1}| \leq |V_n| + Amax\{|W_n|, |V_n|\} + B$, for some given positive constants A and B, and denote $U_n = |W_n| + |V_n|, 0 \leq n \leq N$. Then,$U_n \leq \overline{A^n}U_0 + \overline{B}\frac{\overline{A^n}-1}{\overline{A}-1}$,$0 \leq n \leq N$, where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$.

The proof of Lemma (4.2) follows Lemma 2 of Ming Ma et al [1].
Let $F[t,u,v] = \frac{1}{5}\left[\frac{\underline{k_1^2}+\underline{k_2^2}}{\underline{k_1}+\underline{k_2}}\right] + \frac{4}{5}\left[\frac{\underline{k_2^2}+\underline{k_3^2}}{\underline{k_2}+\underline{k_3}}\right]$,$G[t,u,v] = \frac{1}{5}\left[\frac{\overline{k_1^2}+\overline{k_2^2}}{\overline{k_1}+\overline{k_2}}\right] + \frac{4}{5}\left[\frac{\overline{k_2^2}+\overline{k_3^2}}{\overline{k_2}+\overline{k_3}}\right]$

The domain where F and G are defined is therefore
$K = \{(t,u,v)\backslash 0 \leq t \leq T, -\infty < v < \infty, -\infty < u \leq v\}$.
Theorem 4.1 Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^3\,(K)$ and let the partial derivatives of F and G be bounded over K.Then, for arbitrary fixed $r, 0 \leq r \leq 1$,the approximate solutions given in Eqs (4.1)&(4.2) converge to the exact solutions $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ uniformly in t.

The proof of Theorem (4.1) follows Theorem 1 of Ming Ma et al[1].
Similarly, it can be proved the that approximate solutions given in equations(4.9)&(4.10) converge to the exact solutions$\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ uniformly in t.

5. Numerical Example

Example: 1

Consider the fuzzy initial value problem,

$$y'(t) = y(t), t \in [0,1], \tag{5.1}$$

$$y(0) = (0.8 + 0.125r, 1.1 - 0.1r), 0 < r \leq 1.$$

Case (I)

The exact solution of the equation given in (5.1) under (i)-differentiability is given by

$$Y(t;r) = \begin{bmatrix}(0.8 + 0.125r)e^t + (0.8 + 0.125r)e^{-t}, \\ (1.1 - 0.1r)e^t + (1.1 + 0.1r)e^{-t}\end{bmatrix}, 0 < r \leq 1.$$

At $t = 0.1$ with $h = 0.01$,h=0.001 and h=0.0001, the exact and approximate solutions obtained by the Third Order Runge-Kutta method with Contra Harmonic Mean(RKCHM) and Classical Runge-Kutta(CRK) method are given in the following tables Table1, Table2 and Table3, respectively.

Table: 1

R	Exact $\underline{Y}$	RKCHM $\underline{y}$	CRK $\underline{y}$	Exact $\overline{Y}$	RKCHM $\overline{y}$	CRK $\overline{y}$
0	0.294404204	0.294109519	0.294109517	0.430318295	0.429887877	0.429887897
0.2	0.309099251	0.308789889	0.308789889	0.417830524	0.417412576	0.417412594

0.4	0.323794299	0.323470260	0.323470261	0.405342754	0.404937274	0.40493729
0.6	0.338489346	0.338150630	0.338150633	0.392854983	0.392461973	0.392461986
0.8	0.353184394	0.352831000	0.352831005	0.380367212	0.379986671	0.379986682
1	0.367879441	0.367511370	0.367511378	0.367879441	0.367511370	0.367511378

Table: 2

R	Exact $\underline{Y}$	RKCHM $\underline{y}$	CRK $\underline{y}$	Exact $\overline{Y}$	RKCHM $\overline{y}$	CRK $\overline{y}$
0	0.294404204	0.294109519	0.294109517	0.430318295	0.429887877	0.429887897
0.2	0.309099251	0.308789889	0.308789889	0.417830524	0.417412576	0.417412594
0.4	0.323794299	0.323470260	0.323470261	0.405342754	0.404937274	0.40493729
0.6	0.338489346	0.338150630	0.338150633	0.392854983	0.392461973	0.392461986
0.8	0.353184394	0.352831000	0.352831005	0.380367212	0.379986671	0.379986682
1	0.367879441	0.367511370	0.367511378	0.367879441	0.367511370	0.367511378

Table: 3

R	Exact $\underline{Y}$	RKCHM $\underline{y}$	CRK $\underline{y}$	Exact $\overline{Y}$	RKCHM $\overline{y}$	CRK $\overline{y}$
0	0.294404204	0.294374761	0.294374761	0.430318295	0.430275262	0.430275263
0.2	0.309099251	0.309068339	0.309068339	0.417830524	0.417788740	0.417788740
0.4	0.323794299	0.323761917	0.323761917	0.405342754	0.405302218	0.405302218
0.6	0.338489346	0.338455495	0.338455495	0.392854983	0.392815696	0.392815696
0.8	0.353184394	0.353149073	0.353149073	0.380367212	0.380329174	0.380329174
1	0.367879441	0.367842651	0.367842651	0.367879441	0.367842651	0.367842651

It is found that the proposed Runge-Kutta method has given the best solution. It is clear that the approximate solutions converge to the exact solution as h decreases.

Case (II)

The exact solution of the equation given in (5.1) under (i)-differentiability is given by

$$Y(t;r) = \begin{bmatrix} (-0.025+0.025r)e^{t} + (0.985+0.015r)e^{-t}, \\ (0.025-0.025r)e^{t} + (0.985+0.015r)e^{-t} \end{bmatrix}, 0 < r \leq 1.$$

At $t = 0.1$ with $h = 0.01$,h=0.001 and h=0.0001, the exact and approximate solutions obtained by the Third Order Runge-Kutta method with Contra Harmonic Mean(RKCHM) and

Classical Runge-Kutta(CRK) method are given in the following tables Table 4, Table 5 and Table6, respectively.

Table: 4

R	Exact $\underline{Y}$	RKCHM $\underline{y}$	CRK $\underline{y}$	Exact $\overline{Y}$	RKCHM $\overline{y}$	CRK $\overline{y}$
0	0.353164264	0.349521496	0.349522457	0.371558236	0.367915776	0.367916429
0.2	0.356107299	0.352453470	0.352454401	0.370822477	0.367168894	0.367169578
0.4	0.359050335	0.355385444	0.355386345	0.370086718	0.366422012	0.366422728
0.6	0.361993370	0.358317417	0.358318288	0.369350959	0.365675129	0.365675877
0.8	0.364936406	0.361249390	0.361250232	0.368615200	0.364928246	0.364929026
1	0.367879441	0.364181362	0.364182176	0.367879441	0.364181362	0.364182176

Table: 5

R	Exact $\underline{Y}$	RKCHM $\underline{y}$	CRK $\underline{y}$	Exact $\overline{Y}$	RKCHM $\overline{y}$	CRK $\overline{y}$
0	0.353164264	0.352801711	0.352801721	0.371558236	0.371195687	0.371195693
0.2	0.356107299	0.355743643	0.355743652	0.370822477	0.370458823	0.370458830
0.4	0.359050335	0.358685575	0.358685584	0.370086718	0.369721960	0.369721967
0.6	0.361993370	0.361627506	0.361627515	0.369350959	0.368985096	0.368985104
0.8	0.364936406	0.364569438	0.364569446	0.368615200	0.368248233	0.368248241
1	0.367879441	0.367511370	0.367511378	0.367879441	0.367511370	0.367511378

Table: 6

R	Exact $\underline{Y}$	RKCHM $\underline{y}$	CRK $\underline{y}$	Exact $\overline{Y}$	RKCHM $\overline{y}$	CRK $\overline{y}$
0	0.353164264	0.353128025	0.353128026	0.371558236	0.371521998	0.371521998
0.2	0.356107299	0.356070951	0.356070951	0.370822477	0.370786128	0.370786128
0.4	0.359050335	0.359013876	0.359013876	0.370086718	0.370050259	0.370050259
0.6	0.361993370	0.361956801	0.361956801	0.369350959	0.369314390	0.369314390
0.8	0.364936406	0.364899726	0.364899726	0.368615200	0.368578521	0.368578521
1	0.367879441	0.367842651	0.367842651	0.367879441	0.367842651	0.367842651

6. Conclusion

In this paper, the Third-order Runge-Kutta method with Contra Harmonic Mean is proposed for resolving the Non-autonomous system of ordinary differential equations. This formula is modified for solving the fuzzy differential equations under the generalized differentiability concepts. Numerical example and comparisons with exact solutions reveal that the proposed algorithm is capable of generating precise results.

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