SOLUTION OF PRECONDITIONED SYSTEM OF INTERVAL LINEAR EQUATIONS

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Abstract: In this paper we introduce a simple method based on modified interval arithmetic and the concept of preconditioning for the solution of system of interval linear equations without converting the given system into classical(crisp) linear systems. Numerical examples are provided to show the efficiency of proposed method.

Keywords: Interval arithmetic, Interval matrix, System of interval linear equations, Preconditioning.

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1. Introduction

To model and solve real-world problems, we must deal with uncertainty and vagueness. These uncertainty and vagueness may due to many sources like measurement inaccuracy, simplification of physical models, variations of the parameters of the system, computational errors etc. In such cases, we do not know the exact values of the model parameters, we may know only the possible values of intervals. Interval analysis is an efficient and reliable tool that allows us to handle such uncertainty and vagueness effectively.

Systems of linear equations play a major role in modeling and solving the real world problems. When the model parameters are uncertain and vague, the problem may be modeled as a system of interval linear equations. In the literature, there are methods for computing the smallest interval vector containing the exact solution of the systems of interval linear equations.

Interval arithmetic in matrix computations was introduced by Hansen and Smith [10]. Several authors such as Alfeld and Herzberger [1], Hansen et al [10, 11], Jaulin et al [20], Neumaier [24], Rohn [31, 32, 33], Ganesan and Veeramani [8, 9] etc have studied Interval matrices and systems of interval linear equations. Kolev [18] developed a method for outer interval solution of linear parametric systems. Popova [30] also introduced a method on the solution of parametrised linear systems. Walter Kraemer [35] investigated computing and visualizing solution sets of interval linear systems.

Hansen and Smith [10] proposed the preconditioning method to transform the system into an equivalent system by multiplying the given system by an approximate inverse of the midpoint matrix of $\tilde{A}$. Eugeniusz Zieniuk [7] studied the effectiveness of the existing methods used to solve interval systems of equations generated during the numerical solution of boundary value problems. Jaroslav Horacek and Milan Hladik [14] proposed several methods for computing enclosures of overdetermined interval linear systems. By providing numerical examples, they claimed that their method is better than other methods regarding tightness of enclosures, computation times and other special properties of methods. Milan Hladik [23] proposed a new algorithm, called the magnitude method for solving real preconditioned interval linear equations. Szilvia Huszarszky and Lajos Gergo [34] proposed two methods to bound the solution set of full rank interval linear system of equations based on Hansen's preconditioning and the concept of generalized solution of overdetermined systems of linear equations.

The main objective of this paper is to find the smallest interval vector containing all possible solutions of the system. In this paper, we propose a new method for the solution of the system of interval linear equations using the concept of preconditioning without converting into crisp systems.

The rest of this paper is organized as follows: In Section 2, we extend the Sengupta and Pal's [4] method of comparison of interval numbers to the set of all generalized intervals $\mathbb{D}$. We recall the generalized interval arithmetic on the set of generalized interval numbers $\mathbb{D}$ proposed by Nirmala et.al [26, 27]. In Section 3, we recall the notion of interval matrices, arithmetic operations on interval matrices and also prove some basic theorems for the solution of interval linear systems based on the concept of preconditioning. In Section 4, we propose a simple algorithm for the solution of system of interval linear equations using Intlab. In Section 5, numerical examples are provided and the results are compared with the solutions obtained by other methods.

2. Preliminary Notes

Let $\mathbb{IR} = \{\tilde{a} = [a_1, a_2] : a_1 \leq a_2 \text{ and } a_1, a_2 \in \mathbb{R}\}$ be the set of all proper intervals and $\overline{\mathbb{IR}} = \{\tilde{a} = [a_1, a_2] : a_1 > a_2 \text{ and } a_1, a_2 \in \mathbb{R}\}$ be the set of all improper intervals on the real line $\mathbb{R}$. If $a_1 = a_2 = a$, then $\tilde{a} = [a, a]$ is a real number (or a degenerate interval). We shall use the terms "interval" and "interval number" interchangeably. The mid-point and width (or half-width) of an interval number $\tilde{a} = [a_1, a_2]$ are defined as $m(\tilde{a}) = \left(\frac{a_1 + a_2}{2}\right)$ and $w(\tilde{a}) = \left(\frac{a_2 - a_1}{2}\right)$. We denote the set of generalized intervals (proper and improper) by $\mathbb{D} = \mathbb{IR} \cup \overline{\mathbb{IR}} = \{[a_1, a_2] : a_1, a_2 \in \mathbb{R}\}$. The set of generalized intervals $\mathbb{D}$ is a group with respect to addition and multiplication operations of zero free intervals, while maintaining the inclusion monotonicity.

The "dual" is an important monadic operator proposed by Kaucher [16] that reverses the end-points of the intervals and expresses an element-to-element symmetry between proper and improper intervals in $\mathbb{D}$. For $\tilde{a} = [a_1, a_2] \in \mathbb{D}$, its dual is defined by $\text{dual}(\tilde{a}) = \text{dual}[a_1, a_2] = [a_2, a_1]$. The opposite of an interval $\tilde{a} = [a_1, a_2]$ is $\text{opp}\{[a_1, a_2]\} = [-a_1, -a_2]$ which is the additive inverse of $[a_1, a_2]$ and $\left[\frac{1}{a_1}, \frac{1}{a_2}\right]$ is the multiplicative inverse of $[a_1, a_2]$, provided $0 \notin [a_1, a_2]$.

$$\begin{aligned}
\text{That is} \quad \tilde{a} + (-\text{dual }\tilde{a}) =& \tilde{a} - \text{dual}(\tilde{a}) = [a_1, a_2] - \text{dual}([a_1, a_2]) \\
=& [a_1, a_2] - [a_2, a_1] = [a_1 - a_1, a_2 - a_2] = [0, 0] \\
\text{and} \quad \tilde{a} \times \left(\frac{1}{\text{dual }\tilde{a}}\right) =& [a_1, a_2] \times \left(\frac{1}{\text{dual}([a_1, a_2])}\right) \\
=& [a_1, a_2] \times \frac{1}{[a_2, a_1]} = [a_1, a_2] \times \left[\frac{1}{a_1}, \frac{1}{a_2}\right] = \left[\frac{a_1}{a_1}, \frac{a_2}{a_2}\right] = [1, 1]
\end{aligned}$$

2.1. Comparing Interval Numbers.

Let $\preceq$ be an extended order relation between the interval numbers $\tilde{a} = [a_1, a_2]$, $\tilde{b} = [b_1, b_2] \in \mathbb{D}$, then for $m(\tilde{a}) < m(\tilde{b})$, we construct a premise $(\tilde{a} \preceq \tilde{b})$ which implies that $\tilde{a}$ is inferior to $\tilde{b}$ (or $\tilde{b}$ is superior to $\tilde{a}$.)

An acceptability function $\mathcal{A}_{\preceq} : \mathbb{D} \times \mathbb{D} \longrightarrow [0, \infty)$ is defined as:

$$\mathcal{A}_{\preceq}(\tilde{a}, \tilde{b}) = \mathcal{A}(\tilde{a} \preceq \tilde{b}) = \frac{(m(\tilde{b}) - m(\tilde{a}))}{(w(\tilde{b}) + w(\tilde{a}))},$$

where $w(\tilde{b}) + w(\tilde{a}) \neq 0$. $\mathcal{A}_{\preceq}$ may be interpreted as the grade of acceptability of the first interval number $\tilde{a}$ to be inferior to the second interval number $\tilde{b}$.

2.2. A New Interval Arithmetic.

Ganesan and Veeramani [8] proposed a new interval arithmetic on $\mathbb{IR}$. We extend this arithmetic operations to the set of generalized interval numbers $\mathbb{D}$ and incorporating the concept of dual.

For $\tilde{a} = [a_1, a_2]$, $\tilde{b} = [b_1, b_2] \in \mathbb{D}$ and for $* \in \{+, -, \cdot, \div\}$, we define
$\tilde{a} * \tilde{b} = [m(\tilde{a}) * m(\tilde{b}) - k, m(\tilde{a}) * m(\tilde{b}) + k]$, where
$k = \min\left\{(m(\tilde{a}) * m(\tilde{b})) - \alpha,\ \beta - (m(\tilde{a}) * m(\tilde{b}))\right\}$, α and β are the end points of the interval $\tilde{a} \odot \tilde{b}$ under the existing interval arithmetic. In particular

(i) Addition: $\tilde{a} + \tilde{b} = [a_1, a_2] + [b_1, b_2] = \left[(m(\tilde{a}) + m(\tilde{b})) - k, (m(\tilde{a}) + m(\tilde{b})) + k\right]$, where $k = \left\{\dfrac{(b_2 + a_2) - (b_1 + a_1)}{2}\right\}$.

(ii) Subtraction: $\tilde{a} - \tilde{b} = [a_1, a_2] - [b_1, b_2] = \left[(m(\tilde{a}) - m(\tilde{b})) - k, (m(\tilde{a}) - m(\tilde{b})) + k\right]$, $k = \left\{\dfrac{(b_2 + a_2) - (b_1 + a_1)}{2}\right\}$.

Also if $\tilde{a} = \tilde{b}$, i.e. if $[a_1, a_2] = [b_1, b_2]$, then

$$\tilde{a} - \tilde{b} = \tilde{a} - \text{dual}(\tilde{a}) = [a_1, a_2] - [a_2, a_1] = [a_1 - a_1, a_2 - a_2] = [0, 0] .$$

(iii) Multiplication: $\tilde{a}.\tilde{b} = \tilde{a}\tilde{b} = [a_1, a_2]\,[b_1, b_2] = \left[m(\tilde{a})m(\tilde{b}) - k, m(\tilde{a})m(\tilde{b}) + k\right]$, where $k = \min\left\{(m(\tilde{a})m(\tilde{b})) - \alpha,\ \beta - (m(\tilde{a})m(\tilde{b}))\right\}$, $\alpha = \min(a_1b_1, a_1b_2, a_2b_1, a_2b_2)$ and $\beta = \max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)$.

(iv) Division: $1 \div \tilde{a} = \dfrac{1}{\tilde{a}} = \dfrac{1}{[a_1, a_2]} = \left[\dfrac{1}{m(\tilde{a})} - k, \dfrac{1}{m(\tilde{a})} + k\right]$, where $k = \min\left\{\dfrac{1}{a_2}\left(\dfrac{a_2 - a_1}{a_1 + a_2}\right), \dfrac{1}{a_1}\left(\dfrac{a_2 - a_1}{a_1 + a_2}\right)\right\}$ and $0 \notin [a_1, a_2]$.

Also if $\tilde{a} = \tilde{b}$ i.e. if $[a_1, a_2] = [b_1, b_2]$, then

$$\frac{\tilde{a}}{\tilde{b}} = \frac{\tilde{a}}{\tilde{a}} = \frac{\tilde{a}}{\text{dual}(\tilde{a})} = [a_1, a_2]\,.\frac{1}{[a_2, a_1]} = [a_1, a_2]\,.\left[\frac{1}{a_1}, \frac{1}{a_2}\right] = \left[\frac{a_1}{a_1}, \frac{a_2}{a_2}\right] = [1, 1]$$

From (iii), it is clear that $\lambda\tilde{a} = \begin{cases} [\lambda a_1, \lambda a_2], & \text{for } \lambda \geq 0 \\ [\lambda a_2, \lambda a_1], & \text{for } \lambda < 0. \end{cases}$

3. Main Results

An *interval matrix* $\tilde{A}$ is a matrix whose elements are interval numbers. An interval matrix $\tilde{A}$ will be written as: $\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \cdots & \cdots & \cdots \\ \tilde{a}_{m1} & \cdots & \tilde{a}_{mn} \end{pmatrix} = (\tilde{a}_{ij})_{1 \leq i \leq m,\ 1 \leq j \leq n,}$ where each $\tilde{a}_{ij} = [\underline{a}_{ij},\ \overline{a}_{ij}]$ (or) $\tilde{A} = [\underline{A}, \overline{A}]$ for some $\underline{A}, \overline{A}$ satisfying $\underline{A} \leq \overline{A}$. We use $\mathbb{D}^{m \times n}$ to denote the set of all $(m \times n)$ interval matrices. The midpoint (center) of an interval matrix $\tilde{A}$ is the matrix of midpoints of its interval elements defined as $m(\tilde{A}) = \begin{pmatrix} m(\tilde{a}_{11}) & \cdots & m(\tilde{a}_{1n}) \\ \cdots & \cdots & \cdots \\ m(\tilde{a}_{m1}) & \cdots & m(\tilde{a}_{mn}) \end{pmatrix}$. The width of an interval matrix $\tilde{A}$ is the matrix of widths of its interval elements defined as $w(\tilde{A}) = \begin{pmatrix} w(\tilde{a}_{11}) & \cdots & w(\tilde{a}_{1n}) \\ \cdots & \cdots & \cdots \\ w(\tilde{a}_{m1}) & \cdots & w(\tilde{a}_{mn}) \end{pmatrix}$ which is always nonnegative. We use O to denote the *null matrix* $\begin{pmatrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}$ and $\tilde{O}$ to denote the *null interval matrix* $\begin{pmatrix} \tilde{0} & \cdots & \tilde{0} \\ \cdots & \cdots & \cdots \\ \tilde{0} & \cdots & \tilde{0} \end{pmatrix}$. Also we use I to denote the *identity* *matrix* $\begin{pmatrix} 1 & \cdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \cdots & 1 \end{pmatrix}$ and $\tilde{I}$ to denote the *identity interval matrix* $\begin{pmatrix} \tilde{1} & \cdots & \tilde{0} \\ \cdots & \tilde{1} & \cdots \\ \tilde{0} & \cdots & \tilde{1} \end{pmatrix}$. If $m(\tilde{A}) = m(\tilde{B})$, then the interval matrices $\tilde{A}$ and $\tilde{B}$ are said to be *equivalent* and is denoted by $\tilde{A} \approx \tilde{B}$. In particular if $m(\tilde{A}) = m(\tilde{B})$ and $w(\tilde{A}) = w(\tilde{B})$, then $\tilde{A} = \tilde{B}$. If $m(\tilde{A}) = O$, then we say that $\tilde{A}$ is a *zero interval matrix*. In particular if $m(\tilde{A}) = O$ and $w(\tilde{A}) = O$, then $\tilde{A} = \begin{pmatrix} [0,0] & \ldots & [0,0] \\ \ldots & \ldots & \ldots \\ [0,0] & \ldots & [0,0] \end{pmatrix}$. Also, if $m(\tilde{A}) = O$ and $w(\tilde{A}) \neq O$, then $\tilde{A} = \begin{pmatrix} \tilde{0} & \cdots & \tilde{0} \\ \cdots & \cdots & \cdots \\ \tilde{0} & \cdots & \tilde{0} \end{pmatrix} \approx \tilde{O}$. If $\tilde{A} \not\approx \tilde{O}$ (i.e. $\tilde{A}$ is not equivalent to $\tilde{O}$), then $\tilde{A}$ is said to be a *non-zero interval matrix*. If $m(\tilde{A}) = I$, then we say that $\tilde{A}$ is a *identity interval matrix*. In particular if $m(\tilde{A}) = I$ and $w(\tilde{A}) = O$, then $\tilde{A} = \begin{pmatrix} [1,1] & \cdots & [0,0] \\ \cdots & [1,1] & \cdots \\ [0,0] & \ldots & [1,1] \end{pmatrix}$. Also, if $m(\tilde{A}) = I$ and $w(\tilde{A}) \neq O$, then $\tilde{A} = \begin{pmatrix} \tilde{1} & \cdots & \tilde{0} \\ \cdots & \tilde{1} & \cdots \\ \tilde{0} & \cdots & \tilde{1} \end{pmatrix} \approx \tilde{I}$.

3.1. **Arithmetic Operations on Interval Matrices.** We define the arithmetic operations on interval matrices as follows.

If $\tilde{A}, \tilde{B} \in \mathbb{D}^{m\times n}$, $\tilde{\mathbf{x}} \in \mathbb{D}^n$ and $\tilde{\alpha} \in \mathbb{D}$, then

(i). $\tilde{\alpha}\tilde{A} \approx (\tilde{\alpha}\tilde{a}_{ij})_{1\le i\le m,\ 1\le j\le n}$

(ii). $\tilde{A} + \tilde{B} \approx \left(\tilde{a}_{ij} + \tilde{b}_{ij}\right)_{1\le i\le m,\ 1\le j\le n}$

(iii). $\tilde{A} - \tilde{B} \approx \begin{cases} \left(\tilde{a}_{ij} - \tilde{b}_{ij}\right)_{1\le i\le m,\ 1\le j\le n}, & \text{if } \tilde{A} \not\approx \tilde{B} \\ \tilde{A} - \text{dual}(\tilde{A}) \approx \tilde{O} = O, & \text{if } \tilde{A} \approx \tilde{B} \end{cases}$

(iv). $\tilde{A}\tilde{B} \approx \left(\sum_{k=1}^{n} \tilde{a}_{ik}\tilde{b}_{kj}\right)_{1\le i\le m,\ 1\le j\le n}$

(v). $\tilde{A}\tilde{\mathbf{x}} \approx \left(\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{\mathbf{x}}\right)_{1\le i\le m}$

3.2. **Preconditioning.** A system of interval linear equation can be written as

$$\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}} \tag{1}$$

where $\tilde{A} \in \mathbb{D}^{m\times n}$ be an interval matrix and $\tilde{\mathbf{b}} \in \mathbb{D}^m$ be an interval vector.

Preconditioning is the process of transforming a given interval linear system into a new equivalent linear system. By premultiplying the system (1) with $(m(\tilde{A}))^{-1}$, the system (1) is transformed into a new equivalent interval linear system.

$$\begin{aligned} (m(\tilde{A}))^{-1}\tilde{A} \quad \tilde{\mathbf{x}} &= (m(\tilde{A}))^{-1} \quad \tilde{\mathbf{b}} \\ \Rightarrow \qquad \tilde{C}\tilde{\mathbf{x}} &= \tilde{\mathbf{d}} \end{aligned}$$

in such way that it is easy to find solution. This transformation as preconditioning with the mid point inverse and the non-singular matrix $(m(\tilde{A}))^{-1}$ is the preconditioning matrix of the transformation.

Theorem 3.2.1

Let $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ be a system of m linear equations in n unknowns. If $m(\tilde{A}) \in \mathbb{D}^{m\times n}$ is invertible, then the system $(m(\tilde{A})\,\tilde{A})\tilde{\mathbf{x}} = m(\tilde{A})\,\tilde{\mathbf{b}}$ is equivalent to $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$.

Proof: Let $\tilde{P} = \{\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \,/\, \tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}\}$ be the set of all solution for the system $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$

and $\tilde{Q} = \{\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_n)\, / (m(\tilde{A})\, \tilde{A})\tilde{\mathbf{y}} = m(\tilde{A})\, \tilde{\mathbf{b}}\}$ be the set of all solution for the system $(m(\tilde{A})\, \tilde{A})\tilde{\mathbf{y}} = m(\tilde{A})\, \tilde{\mathbf{b}}$.

$$\text{then} \qquad \begin{aligned} \tilde{\mathbf{r}} \in \tilde{P} &\iff \tilde{A}\tilde{\mathbf{r}} = \tilde{\mathbf{b}} \\ &\iff \tilde{A}\tilde{\mathbf{r}} = (m(\tilde{A}))^{-1}\, (m(\tilde{A}))\, \tilde{\mathbf{b}} \\ &\iff (m(\tilde{A})\, \tilde{A})\tilde{\mathbf{r}} = m(\tilde{A})\, \tilde{\mathbf{b}} \\ &\iff \tilde{\mathbf{r}} \in \tilde{Q} \end{aligned}$$

So, $\tilde{P} \approx \tilde{Q}$. From this, we conclude that the system $(m(\tilde{A})\, \tilde{A})\tilde{\mathbf{x}} = m(\tilde{A})\, \tilde{\mathbf{b}}$ is equivalent to $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$.

4. Algorithm for the solution of system of interval linear equations using IntLab

```
%Solving a system of interval linear
equations by using preconditioning
A=input('Enter the interval matrix: ')
b=input('Enter the vector: ')
format long;
A;b;
n=length(A);
c=mid(A)
d=inv(c)
z=d*A
d2=d*b
%Diagonal matrix
for i=1:n
for k=1:n
if(mid(z(i,k))==0)
z(i,k)=infsup(0,0)
else
z(i,k)
end
z1(i,k)=z(i,k)
end
end
p=[z1 d2]
l=z1
for i=1:n
for k=1:n
if((l(i,k))==(l(i,i))
      && (l(i,k))/(l(i,i)))
l(i,k)=infsup(1,1);
else
l(i,k)=infsup(0,0);
end
s=l;
end
end
q=[s d2]
%Solution of given interval matrix
for i=1:n
 a=inf(z1(i,i));
 b=sup(z1(i,i));
 c=infsup(a,b);
 format long
 c;
m1=(a+b)/2 ;
if(a<=0 && b>=0)
if(m1>0)
m=m1/2;
f=m1-m;
g=m1+m;
h=infsup(f,g);
m2=(f+g)/2;
k1=(1/g)*((g-f)/(g+f));
k2=(1/f)*((g-f)/(g+f));
k=min(k1,k2);
n1=(1/m2)-k ;
n2=(1/m2)+k;
n3=infsup(n1,n2) ;
%Multiplication of two
interval numbers
a=inf(n3);
b=sup(n3);
c=inf(d2(i));
d=sup(d2(i));
format long;
a;b;c;d;
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k=min(k1,k2);
n1=(m1*m2)-k;
```

```
n2=(m1*m2)+k;
hhh=infsup(n1,n2);
infsup(hhh)
else
m3=-m1/2;
f1=m1-m3;
g1=m1+m3;
h1=infsup(f1,g1);
m4=(f1+g1)/2;
k3=(1/g1)*((g1-f1)/(g1+f1));
k4=(1/f1)*((g1-f1)/(g1+f1));
k1=min(k3,k4);
n4=(1/m4)-k1;
n5=(1/m4)+k1;
n6=infsup(n4,n5);
end
% Multiplication of two
interval numbers
a=inf(n6);
b=sup(n6);
c=inf(d2(i));
d=sup(d2(i));
format long;
a;b;c;d;
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k=min(k1,k2);
n1=(m1*m2)-k;
n2=(m1*m2)+k;
hhh=infsup(n1,n2);
infsup(hhh)
else
  k5=(1/b)*((b-a)/(a+b)) ;
  k6=(1/a)*((b-a)/(a+b));
  k7=min(k5,k6);
  n7=(1/m1)-k7 ;
  n8= (1/m1)+k7;
  n9=infsup(n7,n8);
% Multiplication of two
interval numbers
a=inf(n9);
b=sup(n9);
c=inf(d2(i));
d=sup(d2(i));
format long;
a;b;c;d;
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k=min(k1,k2);
n1=(m1*m2)-k;
n2=(m1*m2)+k;
hhh=infsup(n1,n2);
infsup(hhh)
end
end
```

5. Numerical examples

Example 5.1 Consider a system of interval linear equations $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ discussed by Hassan Badry Mohamed A. and El-Owny [13], Alexandre Goldsztejn [3] and Carlos Hoelbig [5] where
$\tilde{A} = \begin{pmatrix} [0.4481568, 0.4498432] & [0.4376422, 0.4393578] \\ [0.4376938, 0.4393062] & [0.6503902, 0.6516098] \end{pmatrix}$ and $\tilde{\mathbf{b}} = \begin{pmatrix} [0.5646710, 0.5667290] \\ [0.6103170, 0.6134830] \end{pmatrix}$

Here, $(m(\tilde{A}))^{-1} = \begin{pmatrix} 6.508909757615598 & -4.384265635506051 \\ -4.384265635506051 & 4.489248050951464 \end{pmatrix}$

By applying the concept of preconditioning, the given system $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ becomes

$$\begin{pmatrix} [0.99097709233703, 1.00902290766297] & [-0.00825686797462, 0.00825686797462] \\ [-0.00731604456254, 0.00731604456254] & [0.99350163347639, 1.00649836652361] \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{x}}_1 \\ \tilde{\mathbf{x}}_2 \end{pmatrix}$$
$$= \begin{pmatrix} [0.98572014687539, 1.01299606815859] \\ [0.25517392336783, 0.27840970137503] \end{pmatrix}$$

That is, the new equivalent system is $\tilde{C}\tilde{\mathbf{x}} \approx \tilde{\mathbf{d}}$. Since $m(C_{12}) = m(C_{21}) = [0, 0]$, $\tilde{C}\tilde{\mathbf{x}} \approx \tilde{\mathbf{d}}$ becomes

$$\Rightarrow \begin{pmatrix} [0.99097709233703, 1.00902290766297] & [0,0] \\ [0,0] & [0.99350163347639, 1.00649836652361] \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{x}_1} \\ \tilde{\mathbf{x}_2} \end{pmatrix}$$
$$= \begin{pmatrix} [0.98572014687539, 1.01299606815859] \\ [0.25517392336783, 0.27840970137503] \end{pmatrix}$$

By applying the concept of dual division, $\tilde{C}\tilde{\mathbf{x}} \approx \tilde{\mathbf{d}}$ becomes

$$\begin{pmatrix} [1,1] & [0,0] \\ [0,0] & [1,1] \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{x}_1} \\ \tilde{\mathbf{x}_2} \end{pmatrix} = \begin{pmatrix} \frac{[0.98572014687539, 1.01299606815859]}{[0.99097709233703, 1.00902290766297]} \\ \frac{[0.25517392336783, 0.27840970137503]}{[0.99350163347639, 1.00649836652361]} \end{pmatrix}$$

Then, the solution set is $\begin{pmatrix} [0.97690561769153, 1.02181059734245] \\ [0.25352641579458, 0.28005720894827] \end{pmatrix}$

Table 1. Comparison of the proposed method with other methods

Proposed Method	**Hassan Badry Mohamed et.al** [13]	**Alexandre Goldsztejn** [2]	**Carlos Hoelbig et.al** [5]
[0.97690561769153, 1.02181059734245]	[0.9745537,1.0242698]	[0.974,1.0246]	[0.9740262,1.02468993]
[0.25352641579458, 0.28005720894827]	[0.2460811,0.2874072]	[0.2458,0.2877]	[0.24574949,0.28783412]

From the table(1), the proposed method yeilds better solution comparing other methods.

Example 5.2 Consider a system of interval linear equations $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ discussed by Hassan Badry Mohamed A. and El-Owny[13] and Carlos Hoelbig[5] where

$$\tilde{A} = \begin{pmatrix} [3,3] & [0,1] & [0,1] \\ [0,1] & [3,3] & [0,1] \\ [0,1] & [0,1] & [3,3] \end{pmatrix} \text{ and } \tilde{\mathbf{b}} = \begin{pmatrix} [1,1] \\ [1,1] \\ [1,1] \end{pmatrix}.$$

Table 2. Comparison of the proposed method with other methods

Proposed Method	**Hassan Badry Mohamed et.al**[13]	**Carlos Hoelbig et.al**[5]
[0.23809523809523, 0.26190476190477]	[0.111111,0.333334]	[0.04394499,0.45605500]
[0.23809523809523, 0.26190476190477]	[0.111111,0.333334]	[0.04394499,0.45605500]
[0.23809523809523, 0.26190476190477]	[0.111111,0.333334]	[0.04394499,0.45605500]

Example 5.3 Consider a system of interval linear equations $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ discussed by Hassan Badry Mohamed A. and El-Owny[13] and Carlos Hoelbig[5] where

$$\tilde{A} = \begin{pmatrix} [1.6, 1.8] & [0.4, 0.5] & [0.2, 0.3] \\ [0,0] & [0.6, 0.8] & [0.2, 0.3] \\ [0.1, 0.1] & [0,0] & [1.4, 1.8] \end{pmatrix} \text{ and } \tilde{\mathbf{b}} = \begin{pmatrix} [1.4, 2] \\ [3, 3.4] \\ [2, 2.3] \end{pmatrix}$$

TABLE 3. Comparison of the proposed method with other methods

Proposed Method	**Hassan Badry Mohamed et.al**[13]	**Carlos Hoelbig et.al**[5]
[-0.54077606354833, -0.02239084463241]	[-1.10487815, 0.33737055]	[-1.08943500, 0.52626809]
[3.29375144108830, 4.87671358328266]	[3.10433362, 5.30254389]	[2.75007232, 5.42039269]
[1.10903572262432, 1.61366220913697]	[1.09236830,1.72177701]	[0.99326595, 1.72943197]

Example 5.4 Consider a system of interval linear equations $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ discussed by Hassan Badry Mohamed A. and El-Owny[13] and Carlos Hoelbig[5] where

$$\tilde{A} = \begin{pmatrix} [1.98, 2.02] & [-1.01, -0.99] & [0, 0] & [0, 0] & [0, 0] \\ [-1.01, -0.99] & [2.97, 3.03] & [-1.01, -0.99] & [0, 0] & [0, 0] \\ [0, 0] & [-1.01, -0.99] & [2.97, 3.03] & [-1.01, -0.99] & [0, 0] \\ [0, 0] & [0, 0] & [-1.01, -0.99] & [2.97, 3.03] & [-1.01, -0.99] \\ [0, 0] & [0, 0] & [0, 0] & [-1.01, -0.99] & [1.98, 2.02] \end{pmatrix}$$

and $\tilde{\mathbf{b}} = \begin{pmatrix} [10, 10] \\ [0, 0] \\ [10, 10] \\ [0, 0] \\ [0, 0] \end{pmatrix}$

TABLE 4. Comparison of the proposed method with other methods

Proposed Method	**Hassan Badry Mohamed et.al**[13]	**Carlos Hoelbig et.al**[5]
[6.98799498297795, 7.19382319884023]	[6.89898004, 7.29765392]	[6.88299293,7.29882524]
[4.10640956971969, 4.25722679391668]	[3.97569664,4.405301763]	[3.95697257,4.40666378]
[5.36193029490616, 5.54716061418475]	[5.26906122,5.656550266]	[5.25126311,5.65782779]
[2.14247455811462, 2.22116180552175]	[2.04981395,2.3273262467]	[2.03528617,2.32835018]
[1.07507615122737, 1.10674203059081]	[1.00461178,1.1871714692]	[0.99390173,1.18791644]

From the above tables (1),(2),(3) and (4), the solution sets obtained by applying the proposed algorithm are sharper (less width) than the solution sets obtained by other techniques[13, 2, 5].

6. Conclusion

We have proposed a new algorithm based on modified interval arithmetic and the duality concept for the solution of system of interval linear equations without converting in to crisp system of linear equations. We have applied the concept of preconditioning to transform the given system into a new equivalent interval linear system which is easier to find the solution. Numerical examples are provided to show the efficiency of the proposed algorithm. Also it can be seen that this preconditioning system gives the better bounds of the solutions of the original system.

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