The Stability Analysis of n-dimensional Quadratic Functional Equations

C. Muthamilarasi[a*] ,G. Balasubramanian[b]
[a]Department of Mathematics, Government Arts College for Men ,Krishnagiri
Pin Code -635001, Tamil Nadu, India.
[a]Department of Mathematics, Government Arts College for Men, Krishnagiri
Pin Code -635001, Tamil Nadu, India.

Emil id: **gbs_geetha@yahoo.com, muthamil22@gmail.com**

Abstract:
In this paper, we prove the Hyers-Ulam stability of the orthogonally n-dimensional quadratic functional equation of the form

$$\nabla\left(rx_1+r^2x_2+r^3x_3\right)+\nabla\left(-rx_1+r^2x_2+r^3x_3\right)+\nabla\left(rx_1-r^2x_2+r^3x_3\right)+\nabla\left(rx_1+r^2x_2-r^3x_3\right)=2\Big[\nabla\left(rx_1+r^2x_2\right)$$
$$+\nabla\left(r^2x_2+r^3x_3\right)+\nabla\left(rx_1+r^3x_3\right)+\nabla\left(rx_1-r^2x_2\right)+\nabla\left(r^2x_2-r^3x_3\right)+\nabla\left(rx_1-r^3x_3\right)\Big]-2[r^4\left(\nabla\left(x_1\right)+f\left(-x_1\right)\right]$$
$$+r^8\left(\nabla\left(x_2\right)+\nabla\left(-x_2\right)\right)+r^{12}\left(\nabla\left(x_3\right)+f\left(-x_3\right)\right)\Big]-\left[r\left(\nabla\left(x_1\right)-\nabla\left(-x_1\right)\right)+r^2\left(\nabla\left(x_2\right)-\nabla\left(-x_2\right)\right)+r^3\left(\nabla\left(x_3\right)-\nabla\left(-x_3\right)\right)\right]$$

and

$$g\left(\sum_{i=1}^{n} x_i\right)+\sum_{j=1}^{n} g\left(-x_j+\sum_{i=1,i\neq j}^{n} x_i\right)=\left(n-3\right)\sum_{1\le i<j\le n} g\left(x_i+x_j\right)+\left(-n^2+5n-2\right)\sum_{i=1}^{n} g\left(x_i\right)$$

where n is a positive integer with $n\geq 3$.

Keywords: Hyers-Ulam stability, Ulam-Gavruta-Rassias stability, Orthogonally Euler-Lagrange functional equation, Quadratic mapping.
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1.Introduction

In 1982, J. M. Rassias[13] followed the innovative approach of the Th. M. Rassias theorem in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\|y\|^p$ for $p,q\in R$ with p + q=1. A generalization of all the above results was obtained by P. Gavruta[5] in 1994 by replacing the unbounded Cauchy difference by a general sontrol function $\phi(x,y)$ in the spirit of Rassias approach.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et al., by considering the summation of both the sum and the product of two p-norms in the sprit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see[1,2,3,4,6,10,12,15] . Some of the functional papers are used to develop this paper which are [5,7,8,9,14].

Definition 1.1A vector space X is called an orthogonally vector space if there is a relation $x\perp y$ on X such that

(i) $x\perp 0,\ 0\perp x$ for all $x\in X$;

(ii) *If* $x\perp y$ and $x,y\neq 0$, then x, y are linearly independent;

(iii) $x\perp y$, then $ax\perp by$ for all $a,b\in\mathbb{R}$;

(iv) *If* P is an two-dimensional subspace of X; then

- (a) for every $x\in P$ there exists $0\neq y\in P$ such that $x\perp y$;
- (b) there exists vectors $x,y\neq 0$ such that $x\perp y$ and $x+y\perp x-y$.

Any vector space can be made into an orthogonally vector space if we define $x \perp 0,\ 0 \perp x$ for all x and for non zero vector x , y define $x \perp y$ iff x, y are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. The pair $(X, \perp)$ is called an orthogonality space. It becomes orthogonality normed space when the orthogonality space equipped with a norm.

In this paper, we investigate n-dimensional Euler-Lagrange functional equation of the form

$$g\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} g\left(-x_j + \sum_{i=1, i\neq j}^{n} x_i\right) = (n-3) \sum_{1\le i<j\le n} g\left(x_i + x_j\right) + \left(-n^2 + 5n - 2\right) \sum_{i=1}^{n} g\left(x_i\right) \quad (1.1)$$

and study the Hyers-Ulam stability in the concept of orthogonality and Ulam-Gavruta-Rassias stability.

A mapping $g : X \to Y$ is called orthogonal quadratic if it satisfies the quadratic functional equation **(1.1)** for all $x_1, x_2, ..., x_n \in X$ with $x_1 \perp x_2 \perp ... \perp x_n$ where X be an orthogonality space and Y be a real banach space.

2. Stability of the Orthogonally Euler-Lagrange Type Functional Equation (1.1) involving Sum of Powers of Norms

In this section, let $(X, \perp)$ denotes an orthogonality normed space with norm $\|.\|_X$ and $(Y, \|.\|_Y)$ is a Banach space.

Theorem 2.1 Let λ and $s(s<2)$ be non-negative real numbers. Let $g : X \to Y$ be a mapping fulfilling

$$\left\| g\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} g\left(-x_j + \sum_{i=1, i\neq j}^{n} x_i\right) - (n-3) \sum_{1\le i<j\le n} g\left(x_i + x_j\right) - \left(-n^2 + 5n - 2\right) \sum_{i=1}^{n} g\left(x_i\right) \right\|_Y$$
$$\le \lambda \left\{ \|x_1\|_X^s + \|x_2\|_X^s + ... + \|x_n\|_X^s \right\} \quad (2.1)$$

for all $x_1, x_2, ..., x_n \in X$ with $x_1 \perp x_2 \perp ... \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A : X \to Y$ such that

$$\|g(x) - A(x)\|_Y \le \frac{\lambda \|x\|_X^s}{2(n-5)(2^2 - 2^s)} \quad (2.2)$$

for all $x \in X$.

Proof. Setting $(x_1, x_2, ..., x_n)$ by $(0, 0, ..., 0)$ in (2.1), we have $f(0) = 0$. Letting Replacing $(x_1, x_2, ..., x_n)$ by $(x, -x, x, -x, x, 0, ..., 0)$ in (2.1), we obtain

$$\|2(n-5) g(2x) - 8(n-5) g(x)\|_Y \le \lambda \left\{ \|x\|_X^P \right\} \quad (2.3)$$

for all $x \in X$. Since $x \perp 0$, we get

$$\left\| \frac{g(2x)}{4} - g(x) \right\|_Y \le \frac{1}{8(n-5)} \lambda \left\{ \|x\|_X^P \right\} \quad (2.4)$$

for all $x \in X$. Now replacing x *by* $2x$ and divided by 4 in (2.4) and summing the resulting inequality with (2.4), we obtain

$$\left\| \frac{g(2^2 x)}{2^4} - g(x) \right\|_Y \le \frac{\lambda}{8(n-5)} \left\{ 1 + \frac{2^s}{2^2} \right\} \|x\|_X^s \quad (2.5)$$

for all $x \in X$. Using the induction on n, we receive

$$\left\|\frac{g(2^n x)}{2^{2n}} - g(x)\right\|_Y \le \frac{\lambda}{8(n-5)}\sum_{k=0}^{n-1}\frac{2^{sk}}{2^{2k}}\|x\|_X^s \le \frac{\lambda}{8(n-5)}\sum_{k=0}^{\infty}\frac{2^{sk}}{2^{2k}}\|x\|_X^s \tag{2.6}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{\frac{g(2^n x)}{2^{2n}}\right\}$ replace x *by* $2^m x$ and divided by 2^{2m} in (2.6), for any $n, m > 0$ we get

$$\begin{aligned}\left\|\frac{g(2^{n+m} x)}{2^{2(n+m)}} - \frac{g(2^m x)}{2^{2m}}\right\|_Y &= \frac{1}{2^{2m}}\left\|\frac{g(2^{n+m} x)}{2^{2n}} - g(2^m x)\right\|_Y \\ &\le \frac{\lambda}{8(n-5)2^{2m}}\sum_{k=0}^{n-1}\frac{2^{sk}}{2^{2k}}\|2^m x\|_X^s \\ &\le \frac{\lambda}{8(n-5)}\sum_{k=0}^{\infty}\frac{2^{s(k+m)}}{2^{2(k+m)}}\|x\|_X^s \\ &\le \frac{\lambda}{8(n-5)}\sum_{k=0}^{\infty}\frac{1}{2^{(2-s)(k+m)}}\|x\|_X^s \qquad (2.7)\end{aligned}$$

As $s < 2$, the R.H.S. of (2.7) tends to 0 as $m \to \infty$ for all $x \in X$. Thus $\left\{\frac{g(2^n x)}{2^{2n}}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \to Y$ such that

$$A(x) = \lim_{n\to\infty}\frac{g(2^n x)}{2^{2n}}, \ \forall \ x \in X.$$

Taking $n \to \infty$ in (2.6), we have at the formula (2.2) for all $x \in X$. To prove A satisfies (1.1), replace $(x_1, x_2, ..., x_n)$ by $(2^n x_1, 2^n x_2, ..., 2^n x_n)$ in (2.1) and divided by 2^{2n} then it follows that

$$\frac{1}{2^{2n}}\left\| g\left(\sum_{i=1}^{n} 2^n x_i\right) + \sum_{j=1}^{n} g\left(2^n\left(-x_j + \sum_{i=1, i\ne j}^{n} x_i\right)\right) - (n-3)\sum_{1\le i<j\le n} g\left(2^n\left(x_i + x_j\right)\right) - \left(-n^2+5n-2\right)\sum_{i=1}^{n} g\left(2^n x_i\right)\right\|_Y$$
$$\le \frac{\lambda}{2^{2n}}\left\{\|2^n x_1\|_X^s + \|2^n x_2\|_X^s + ..., + \|2^n x_n\|_X^s\right\}.$$

Again taking limit as $n \to \infty$ in the above inequality, we have

$$\left\| A\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} A\left(-x_j + \sum_{i=1, i\ne j}^{n} x_i\right) - (n-3)\sum_{1\le i<j\le n} A\left(x_i + x_j\right) - \left(-n^2+5n-2\right)\sum_{i=1}^{n} A\left(x_i\right)\right\|_Y = 0$$

which gives

$$g\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} g\left(-x_j + \sum_{i=1, i\ne j}^{n} x_i\right) = (n-3)\sum_{1\le i<j\le n} g\left(x_i + x_j\right) + \left(-n^2+5n-2\right)\sum_{i=1}^{n} g\left(x_i\right)$$

for all $x_1, x_2, ..., x_n \in X$ with $x_1 \perp x_2 \perp ... \perp x_n$. Therefore $A: X \to Y$ is an orthogonally quadratic mapping which satisfies (1.1).

To prove the uniqueness of A. Let B be another quadratic function satisfying (1.1) and the inequality (2.2). Then

$$\begin{aligned}\|A(x)-B(x)\|_Y &= \frac{1}{2^{2n}}\left\|A\left(2^n x\right)-B\left(2^n x\right)\right\|_Y \\ &\le \frac{1}{2^{2n}}\left(\left\|A\left(2^n x\right)-g\left(2^n x\right)\right\|_Y+\left\|g\left(2^n x\right)-B\left(2^n x\right)\right\|_Y\right) \\ &\le \frac{\lambda}{8(n-5)}\sum_{k=0}^{\infty}\frac{1}{2^{(2-s)(k+m)}}\|x\|_X^s \quad \to 0 \text{ as } n\to\infty\end{aligned}$$

for all $x \in X$. Therefore A is unique. This completes the proof of the theorem.

Theorem 2.2 Let λ and $s(s>2)$ be non-negative real numbers. Let $g: X \to Y$ be a mapping fulfilling

$$\left\|g\left(\sum_{i=1}^{n} x_i\right)+\sum_{j=1}^{n} g\left(-x_j+\sum_{i=1, i\neq j}^{n} x_i\right)-(n-3)\sum_{1\le i<j\le n} g\left(x_i+x_j\right)-\left(-n^2+5n-2\right)\sum_{i=1}^{n} g\left(x_i\right)\right\|_Y$$

$$\le \lambda\left\{\|x_1\|_X^s+\|x_2\|_X^s+\ldots+\|x_n\|_X^s\right\} \qquad (2.8)$$

for all $x_1, x_2, \ldots, x_n \in X$ with $x_1 \perp x_2 \perp \ldots \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \to Y$ such that

$$\|g(x)-A(x)\|_Y \le \frac{\lambda\|x\|_X^s}{2(n-5)\left(2^s-2^2\right)} \qquad (2.9)$$

for all $x \in X$.

Proof. Replacing x *by* $\frac{x}{2}$ in (2.3), we arrive

$$\left\|4g\left(\frac{x}{2}\right)-g(x)\right\|_Y \le \frac{1}{2(n-5)2^s}\lambda\left\{\|x\|_X^s\right\} \qquad (2.10)$$

for all $x \in X$. Now replacing x *by* $\frac{x}{2}$ and multiply by 4 in (2.10), we obtain

$$\left\|4g\left(\frac{x}{2}\right)-2^4 g\left(\frac{x}{2^2}\right)\right\|_Y \le \frac{\lambda}{2(n-5)2^{2(s-1)}}\|x\|_X^s \qquad (2.11)$$

From (2.10) and (2.11), we get

$$\left\|g(x)-2^4 g\left(\frac{x}{2^2}\right)\right\|_Y \le \frac{\lambda}{2(n-5)2^s}\left\{1+\frac{1}{2^{s-2}}\right\}\|x\|_X^s \qquad (2.12)$$

for all $x \in X$. Using the induction on n, we receive

$$\left\|2^{2n} g\left(\frac{x}{2^n}\right)-g(x)\right\|_Y \le \frac{\lambda}{2(n-5)2^s}\sum_{k=0}^{n-1}\frac{2^{2k}}{2^{sk}}\|x\|_X^s \le \frac{\lambda}{2(n-5)2^s}\sum_{k=0}^{\infty}\frac{2^{2k}}{2^{sk}}\|x\|_X^s \qquad (2.13)$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{2^{2n} g\left(\frac{x}{2^n}\right)\right\}$ replace x *by* $\frac{x}{2^m}$ and multiply by 2^{2m} in (2.13), for any $n, m > 0$ we get

$$\begin{aligned}\left\|2^{2(n+m)} g\left(\frac{x}{2^{n+m}}\right)-2^{2m} g\left(\frac{x}{2^m}\right)\right\|_Y &= 2^{2m}\left\|2^{2n} g\left(\frac{x}{2^{n+m}}\right)-g\left(\frac{x}{2^m}\right)\right\|_Y \\ &\le \frac{2^{2m}\lambda}{2(n-5)\,2^s}\sum_{k=0}^{n-1}\frac{2^{2k}}{2^{sk}}\left\|\frac{x}{2^m}\right\|_X^s \qquad (2.14)\end{aligned}$$

As $s>2$, the R.H.S. of (2.14) tends to 0 as $m\to\infty$ for all $x\in X$. Thus $\left\{2^{2n}g\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A:X\to Y$ such that

$$A(x)=\lim_{n\to\infty}2^{2n}g\left(\frac{x}{2^n}\right),\ \forall\ x\in X.$$

Taking $n\to\infty$ in (2.13), we have at the formula (2.9) for all $x\in X$. To prove the uniqueness of A and it satisfies the equation (1.1), the proof is similar to that of Theorem 2.1

3. Stability of the Orthogonally Euler-Lagrange Type Functional Equation (1.1) involving Constant only.

Theorem 3.1 Let λ be non-negative real number. Let $g:X\to Y$ be a mapping fulfilling

$$\left\|g\left(\sum_{i=1}^{n}x_i\right)+\sum_{j=1}^{n}g\left(-x_j+\sum_{i=1,i\neq j}^{n}x_i\right)-(n-3)\sum_{1\le i<j\le n}g\left(x_i+x_j\right)-\left(-n^2+5n-2\right)\sum_{i=1}^{n}g\left(x_i\right)\right\|_Y\le\lambda \quad (3.1)$$

for all $x_1,x_2,...,x_n\in X$ with $x_1\perp x_2\perp...\perp x_n$. Then there exists a unique orthogonally quadratic mapping $A:X\to Y$ such that

$$\left\|g(x)-A(x)\right\|_Y\le\frac{\lambda}{6(n-5)} \quad (3.2)$$

for all $x\in X$.

Proof. Setting $(x_1,x_2,...,x_n)$ by $(0,0,...,0)$ in (3.1), we have $f(0)=0$. Replacing $(x_1,x_2,...,x_n)$ by $(x,-x,x,-x,x,0,...,0)$ in (3.1), we obtain

$$\left\|2(n-5)g(2x)-8(n-5)g(x)\right\|_Y\le\lambda \quad (3.3)$$

for all $x\in X$. Since $x\perp 0$, we get

$$\left\|\frac{g(2x)}{4}-g(x)\right\|_Y\le\frac{1}{8(n-5)}\lambda \quad (3.4)$$

for all $x\in X$. Now replacing x *by* $2x$ and divided by 4 in (3.4) and summing the resulting inequality with (3.4), we obtain

$$\left\|\frac{g(2^2x)}{2^4}-g(x)\right\|_Y\le\frac{\lambda}{8(n-5)}\left\{1+\frac{1}{2^2}\right\} \quad (3.5)$$

for all $x\in X$. Using the induction on n, we receive

$$\left\|\frac{g(2^nx)}{2^{2n}}-g(x)\right\|_Y\le\frac{\lambda}{8(n-5)}\sum_{k=0}^{n-1}\frac{1}{2^{2k}}\le\frac{\lambda}{8(n-5)}\sum_{k=0}^{\infty}\frac{1}{2^{2k}} \quad (3.6)$$

for all $x\in X$. In order to prove the convergence of the sequence $\left\{\frac{g(2^nx)}{2^{2n}}\right\}$ replace x *by* 2^mx and divided by 2^{2m} in (3.6), for any $n,m>0$ we get

$$\left\|\frac{g(2^{n+m}x)}{2^{2(n+m)}}-\frac{g(2^mx)}{2^{2m}}\right\|_Y=\frac{1}{2^{2m}}\left\|\frac{g(2^{n+m}x)}{2^{2n}}-g(2^mx)\right\|_Y$$

$$\le\frac{\lambda}{8(n-5)2^{2m}}\sum_{k=0}^{n-1}\frac{1}{2^{2k}}$$

$$\leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{\infty} \frac{1}{2^{2(k+m)}} \tag{3.7}$$

As $s < 2$, the R.H.S. of (3.7) tends to 0 as $m \to \infty$ for all $x \in X$. Thus $\left\{ \frac{g(2^n x)}{2^{2n}} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \to Y$ such that

$$A(x) = \lim_{n\to\infty} \frac{g(2^n x)}{2^{2n}}, \ \forall \ x \in X.$$

Taking $n \to \infty$ in (3.6), we have at the formula (3.2) for all $x \in X$. To prove A satisfies (1.1), replace $(x_1, x_2, ..., x_n)$ by $(2^n x_1, 2^n x_2, ..., 2^n x_n)$ in (3.1) and divided by 2^{2n} then it follows that

$$\frac{1}{2^{2n}} \left\| g\left(\sum_{i=1}^{n} 2^n x_i \right) + \sum_{j=1}^{n} g\left(2^n \left(-x_j + \sum_{i=1, i\neq j}^{n} x_i \right) \right) - (n-3) \sum_{1\leq i<j\leq n} g\left(2^n (x_i + x_j) \right) - \left(-n^2 + 5n - 2 \right) \sum_{i=1}^{n} g\left(2^n x_i \right) \right\|_Y \leq \frac{\lambda}{2^{2n}}$$

Again taking limit as $n \to \infty$ in the above inequality, we have

$$\left\| A\left(\sum_{i=1}^{n} x_i \right) + \sum_{j=1}^{n} A\left(-x_j + \sum_{i=1, i\neq j}^{n} x_i \right) - (n-3) \sum_{1\leq i<j\leq n} A(x_i + x_j) - \left(-n^2 + 5n - 2 \right) \sum_{i=1}^{n} A(x_i) \right\|_Y = 0$$

which gives

$$g\left(\sum_{i=1}^{n} x_i \right) + \sum_{j=1}^{n} g\left(-x_j + \sum_{i=1, i\neq j}^{n} x_i \right) = (n-3) \sum_{1\leq i<j\leq n} g(x_i + x_j) + \left(-n^2 + 5n - 2 \right) \sum_{i=1}^{n} g(x_i)$$

for all $x_1, x_2, ..., x_n \in X$ with $x_1 \perp x_2 \perp ... \perp x_n$. Therefore $A: X \to Y$ is an orthogonally quadratic mapping which satisfies (1.1).

To prove the uniqueness of A. Let B be another quadratic function satisfying (1.1) and the inequality (3.2). Then

$$\begin{aligned} \|A(x) - B(x)\|_Y &= \frac{1}{2^{2n}} \left\| A(2^n x) - B(2^n x) \right\|_Y \\ &\leq \frac{1}{2^{2n}} \left(\left\| A(2^n x) - g(2^n x) \right\|_Y + \left\| g(2^n x) - B(2^n x) \right\|_Y \right) \\ &\leq \frac{\lambda}{8(n-5)} \sum_{k=0}^{\infty} \frac{1}{2^{(2-s)(k+m)}} \qquad \to 0 \ as \ n \to \infty \end{aligned}$$

for all $x \in X$. Therefore A is unique. This completes the proof of the theorem.

Theorem 3.2 Let λ be non-negative real numbers. Let $g: X \to Y$ be a mapping fulfilling

$$\left\| g\left(\sum_{i=1}^{n} x_i \right) + \sum_{j=1}^{n} g\left(-x_j + \sum_{i=1, i\neq j}^{n} x_i \right) - (n-3) \sum_{1\leq i<j\leq n} g(x_i + x_j) - \left(-n^2 + 5n - 2 \right) \sum_{i=1}^{n} g(x_i) \right\|_Y \leq \lambda \tag{3.8}$$

for all $x_1, x_2, ..., x_n \in X$ with $x_1 \perp x_2 \perp ... \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \to Y$ such that

$$\|g(x) - A(x)\|_Y \leq \frac{\lambda}{-6(n-5)} \tag{3.9}$$

for all $x \in X$.

Proof. Replacing x *by* $\frac{x}{2}$ in (3.3), we arrive

$$\left\| 4g\left(\frac{x}{2}\right) - g(x) \right\|_Y \le \frac{1}{2(n-5)}\lambda \tag{3.10}$$

for all $x \in X$. Now replacing x *by* $\frac{x}{2}$ and multiply by 4 in (3.10), we obtain

$$\left\| 4g\left(\frac{x}{2}\right) - 2^4 g\left(\frac{x}{2^2}\right) \right\|_Y \le \frac{2^2\lambda}{2(n-5)} \tag{3.11}$$

From (3.10) and (3.11), we get

$$\left\| g(x) - 2^4 g\left(\frac{x}{2^2}\right) \right\|_Y \le \frac{\lambda}{2(n-5)}\left\{1 + \frac{1}{2^{-2}}\right\} \tag{3.12}$$

for all $x \in X$. Using the induction on n, we receive

$$\left\| 2^{2n} g\left(\frac{x}{2^n}\right) - g(x) \right\|_Y \le \frac{\lambda}{2(n-5)}\sum_{k=0}^{n-1} 2^{2k} \le \frac{\lambda}{2(n-5)}\sum_{k=0}^{\infty} 2^{2k} \tag{3.13}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{2^{2n} g\left(\frac{x}{2^n}\right)\right\}$ replace x *by* $\frac{x}{2^m}$ and multiply by 2^{2m} in (3.13), for any $n, m > 0$ we get

$$\left\| 2^{2(n+m)} g\left(\frac{x}{2^{n+m}}\right) - 2^{2m} g\left(\frac{x}{2^m}\right) \right\|_Y = 2^{2m}\left\| 2^{2n} g\left(\frac{x}{2^{n+m}}\right) - g\left(\frac{x}{2^m}\right) \right\|_Y$$

$$\le \frac{2^{2m}\lambda}{2(n-5)}\sum_{k=0}^{n-1} 2^{2k} \tag{3.14}$$

As $s > 2$, the R.H.S. of (3.14) tends to 0 as $m \to \infty$ for all $x \in X$. Thus $\left\{2^{2n} g\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A: X \to Y$ such that

$$A(x) = \lim_{n\to\infty} 2^{2n} g\left(\frac{x}{2^n}\right), \ \forall\ x \in X.$$

Taking $n \to \infty$ in (3.13), we have at the formula (3.9) for all $x \in X$. To prove the uniqueness of A and it satisfies the equation (1.1), the proof is similar to that of Theorem 3.1

4. Stability of the Orthogonally Euler-Lagrange Type Functional Equation (1.1) involving Sum of the Product of Powers of Norms

Theorem 4.1 Let λ and $s\left(s < \frac{2}{n}\right)$ be non-negative real numbers. Let $g: X \to Y$ be a mapping fulfilling

$$\left\| g\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} g\left(-x_j + \sum_{i=1, i\ne j}^{n} x_i\right) - (n-3)\sum_{1\le i<j\le n} g\left(x_i + x_j\right) - \left(-n^2 + 5n - 2\right)\sum_{i=1}^{n} g\left(x_i\right) \right\|_Y$$

$$\le \lambda\left\{\left(\|x_1\|_X^s \|x_2\|_X^s \ldots \|x_n\|_X^s\right) + \left(\|x_1\|_X^{ns} + \|x_2\|_X^{ns} + \ldots + \|x_n\|_X^{ns}\right)\right\}$$

for all $x_1, x_2, \ldots, x_n \in X$ with $x_1 \perp x_2 \perp \ldots \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \to Y$ such that

$$\left\| g(x) - A(x) \right\|_Y \le \frac{\lambda \|x\|_X^{ns}}{2(n-5)(2^2 - 2^{ns})}$$

for all $x \in X$.

Proof. The proof is same as the proof of Theorem 2.1.

Theorem 4.2 Let λ and $s\left(s > \frac{2}{n}\right)$ be non-negative real numbers. Let $g: X \to Y$ be a mapping fulfilling

$$\left\| g\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} g\left(-x_j + \sum_{i=1, i\ne j}^{n} x_i\right) - (n-3) \sum_{1\le i<j\le n} g\left(x_i + x_j\right) - \left(-n^2 + 5n - 2\right) \sum_{i=1}^{n} g\left(x_i\right) \right\|_Y$$
$$\le \lambda \left\{ \left(\|x_1\|_X^s \|x_2\|_X^s \dots \|x_n\|_X^s \right) + \left(\|x_1\|_X^{ns} + \|x_2\|_X^{ns} + \dots + \|x_n\|_X^{ns} \right) \right\}$$

for all $x_1, x_2, ..., x_n \in X$ with $x_1 \perp x_2 \perp ... \perp x_n$. Then there exists a unique orthogonally quadratic mapping $A: X \to Y$ such that

$$\left\| g(x) - A(x) \right\|_Y \le \frac{\lambda \|x\|_X^{ns}}{2(n-5)(2^{ns} - 2^2)}$$

for all $x \in X$.

Proof. The proof is same as the proof of Theorem 2.2.

References

[1] J. Alonso and C. Benitez, Orthogonality in normed linear spaces: survey I. main properties, Extracta Math., Vol. 3, (1988), 1-15.

[2] J. Alonso and C. Benitez, Orthogonality in normed linear spaces: a survey II. Relations between main orthogonalities, Extracta Math., Vol. 4, (1989), 121-131.

[3] F. Drjevic, On a functional which is quadratic on a orthogonal vectors, Publ. Inst. Math., Vol. 54, (1986), 63-71.

[4] M. Fochi, Functional equations in a orthogonal vectors, Aequationes Math., Vol. 38, (1989), 28-40.

[5] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., Vol. 184, (1994), 431-436.

[6] R. Ger and J. Sikorska, Stability of the orthogonal additivity, Bull. Polish Acad. Sci. Math., Vol. 43, (1995), 143-151.

[7] V.Govindan, S.Murthy, M.Saravanan, Generalized Ulam-HyersStability of two types of n-dimensional Quadratic functional equationin Banach Space: Direct and Fixed Point Methods. Int. J. Math Trends and its Tech., Vol. 51(6), (2017), 396-403

[8] V.Govindan, S.Murthy, G.Kokila, Fixed point and stability of icosic functional equation in quasi beta normed spaces, Malaya J. mathematic, Vol. 6(1),(2018), 261-275

[9] V. Govindan and K. Tamilvanan, Stability of Functional Equation in Banach Space: Using Two Different Methods, Int. J. Math. And Appl., Vol. 6(1-C), (2018), 527-536.

[10] S. Gudder and D. Strawther, Orthogonally additive and orthogonally increasing functions on vector spaces, Pacific J. Math. Vol. 58, (1995), 427-436.

[11] L. Paganoni and J. Ratz, Conditional functional equations and orthogonal additivity, Aequationes math., Vol. 50, (1995), 135-142.

[12] C. Park and Th. M. Rassias, Fixed points and stability of the Cauchy functional equation, Australian J. Math. Anal. Appl., Vol. 6(1), (2009), 1-9.

[13] J. M. Rassias, On approximately of approximately linear mappings by linear mappings, J. Funct. Anal., USA, Vol. 46, (1982), 126-130.

[14] J. M. Rassias, On the stability of the non-linear Euler-Lagrange functional equation in real normed linear spaces, J. Math. Phys. Sci., Vol. 28, (1994), 231-235.
[15] J. M. Rassias, On the stability of the general Euler-Lagrange functional equation, Demonstratio Math., Vol. 29, (1996), 755-766.
[16] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., Vol. 72, (1978), 297-300.