Existence and multiplicity of solutions for Nonlocal elliptic problem involving $p-$biharmonic operator and concave-convex nonlinearities

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Abstract

This article is devoted to the study of the existence and multiplicity of weak solutions for a p-biharmonic Kirchhoff type problem involving convex and concave nonlinearities which may change sign on domain Ω. This type of problem is appearing in numerous physical phenomena such as systems of particles in thermodynamical equilibrium, dielectric breakdown, image restoration and biological phenomena etc. By means of variational methods, the multiplicity of the weak solutions to considered problem is obtained.

Keywords. Existence; Ekeland variational principle; $p-$Biharmonic Kirchhoff-type equation; Nonlocal problems.

AMS subject classification. 65N30, 49M99, 65H05.

1 Introduction

The purpose of this article is to investigate the existence and multiplicity of positive solutions for the following problem involving concave and convex nonlinearities:

$$\begin{cases} \left(M \int_\Omega |\Delta u|^p dx\right) \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & in \;\; x \in \Omega, \\ u(x) = \nabla u(x) = 0 & on \;\; x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is the complement of a smooth bounded domain D in $\mathbb{R}^N(N \geq 3)$ with boundary $\partial\Omega$, $\lambda > 0$, $M(s) = a + bs^k$, $a, b > 0$, $k \geq 0$, $f(x)$ and $g(x)$ are continuous functions which may change sign on Ω. The parameters p, q, r satisfy $1 < q < p(k+1) < r < p^* = \dfrac{Np}{N-p}$ and the operator $\Delta(|\Delta u|^{p-2}\Delta u)$ is the $p-$biharmonic with $1 < p < N$. Problem (1) is called a nonlocal one because of the presence of an integral in Ω in (1), the equation is not a point wise identity and therefore it is called a nonlocal problem. The nonlocal property of the problem of interest generates a lot of difficulties, which makes the study of such a problem particularly interesting. The nonlocal differential equations are also known as Kirchhoff type equations have received specific attention in recent years, an important number of surveys dealing with this type of equations can model phenomenons which arise from the study of electrorheological fluids or elastic mechanics, in numerous physical phenomena such as systems of particles in thermodynamical equilibrium, dielectric breakdown, image restoration and biological phenomena. We may refer to [2, 3, 4] as important references in this field. In 1983, Kirchhoff introduced the following general version of the Kirchhoff equation (see [5])

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0 \qquad (2)$$

that extends the classical D′Alembert′s wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in equation (2) have the following meanings: L is the length of the string, h is the area of cross-section,E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. The considered proplem becomes an object of a large interest after J. L. Lions an abstract framework analysis in [6]. Recently, C. Y. Chen et al. in [1] have investigated the multiplicity of solutions to a class of Dirichlet boundary value problems by using Nehari manifold and fibering maps for the case $p = 2$. The study of Kirchhoff type equations has already been extended to the case involving $p-$Laplacian operator (or details, see [9, 13]) and $p(x)-$Laplacian operator (see [10-12, 14]). Motivated by these findings, we now extend the analysis to the $p-$biharmonic Kirchhoff type equation of (1) with the unbounded domain Ω. In place of Nehari manifold methods, we will use Ekeland′s variational principle and the Mountain Pass Theorem to study the existence of multiple solutions for problem (1). Note that the weak solutions of problem (1) are considered in the generalized Sobolev space

$$X = W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega).$$

Through out this paper,we make the following assumptions:

(A1) $M(s) = a + bs^k$, $a, b > 0$, $0 \le k < \dfrac{p}{N-p}, \forall s \ge 0$

(A2) $f(x) \in L^{q_0}(\Omega)\cap L^\infty(\Omega)$, $g(x) \in L^{r_0}(\Omega)\cap L^\infty(\Omega)$ with $q_0 = \dfrac{p^*}{p^*-q}$ and $r_0 = \dfrac{p^*}{p^*-r}$. Also, $g(x) > 0$ in non-empty open domain $\Omega_2 \subset \Omega$.

(A3) $1 < q < p < p(k + 1) < r < p^* = \dfrac{Np}{N-p}$

Our main result is given as follows.

Theorem 1 Suppose that (A1)-(A3) hold true. Then there exist $\wedge_0 > 0$ such that for all $\lambda \in (0, \wedge_0)$, problem (1) admits atleast two nontrivial weak solutions in X.

2 Proof of Theorem 1

We discuss the existence of two non-trivial weak solutions of (1) by using the mountain pass theorem of Ambrosetti and Rabinowitz and Ekeland′s variational principle. The corresponding Euler lagrange functional associated to (1) is given by

$$I(u)=\frac{a}{p}\int_\Omega|\Delta u|^p dx+\frac{b}{p(k+1)}\left(\int_\Omega|\Delta u|^p\right)^{k+1}-\frac{\lambda}{q}\int_\Omega f(x)|u|^q dx-\frac{1}{r}\int_\Omega g(x)|u|^r dx=0, \qquad (3)$$

It is easy to verify that $I \in C^1(X,\mathbb{R})$ with norm $\|u\| = \left(\int_\Omega |\Delta u|^p dx\right)^{1/p}$ is weakly lower semi-continuous with the derivative given by

$$\langle I'(u), v\rangle = a\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v dx + b\left(\int_\Omega |\Delta u|^p\right)^k \int_\Omega |\Delta u|^{p-2}\Delta u\Delta v$$
$$-\lambda\int_\Omega f(x)|u|^{q-2}u dx - \int_\Omega g(x)|u|^{r-2}u dx = 0, \quad (4)$$

for all $u, v \in X$. Thus, we notice that we can seek weak solutions of (3) as critical point of the energetic functional I.

Throughout the article, the letters $c, c_1, C_i, i = 1, 2, ...$ denote positive constants which may change from line to line.

Lemma 1 Assume that (H1)- (H3) hold. There exists $\wedge_0 > 0$ such that for all $\lambda \in (0, \wedge_0)$, there exist $\rho > 0$ and $\alpha > 0$ such that $I(u) \geq \alpha > 0$ forany $u \in X$ with $\|u\| = \rho$.
Proof By the Holder′s inequality, it follows from (A2) that

$$|\frac{\lambda}{q}\int_\Omega f(x)|u|^q dx| \leq \frac{\lambda}{q}\|f\|_{q_0}\|u\|^q_{p^*} \leq \frac{\lambda}{q}S^{-q/p}\|f\|_{q_0}\|u\|^q \quad (5)$$

Similarly,

$$|\frac{1}{r}\int_\Omega g(x)|u|^r dx| \leq \frac{1}{r}\|f\|_{r_0}\|u\|^r_{p^*} \leq \frac{S^{-r/p}}{r}\|g\|_{r_0}\|u\|^r \quad (6)$$

Thus,

$$I(u) \geq \frac{a}{p}\|u\|^p + \frac{b}{p(k+1)}\|u\|^{p(k+1)} - \frac{\lambda}{q}S^{-q/p}\|f\|_{q_0}\|u\|^q - \frac{S^{-r/p}}{r}\|g\|_{r_0}\|u\|^r,$$
$$I(u) \geq \frac{a}{p}\|u\|^p + \frac{b}{p(k+1)}\|u\|^{p(k+1)} - \lambda C_1\|u\|^q - C_2\|u\|^r,$$
$$I(u) \geq \frac{b}{p(k+1)}\|u\|^{p(k+1)} - \lambda C_1\|u\|^q - C_2\|u\|^r,$$
$$I(u) \geq \frac{b}{p(k+1)} - \lambda C_1\|u\|^{q-p(k+1)} - C_2\|u\|^{r-p(k+1)}, \quad (7)$$

Let us define

$$h(t) = \lambda C_1 t^{q-p(k+1)} - C_2 t^{r-p(k+1)}, t > 0 \quad (8)$$

Since the function $h(t) \to +\infty$ whenever $t \to 0^+$ or $t \to +\infty$. So that the function $h(t)$ minimizes at some point $t_1 > 0$. In order to find t_1, we obtain the $h'(t)$ at point t_1.
so that

$$t_1 = \frac{\lambda C_1(p(k+1)-q)}{C_2(r-p(k+1))}^{1/(r-q)} \equiv \lambda^{1/(r-q)}C_3^{1/(r-q)} > 0. \quad (9)$$

Futhermore $h(t_1) < b/p(k+1)$ implies that

$$h(t_1) = C_1(r-q)(r-m)^{-1}C_3^{(q-m)(r-q)} < b/p(k+1). \tag{10}$$

Therefore,it follows from (7) and (10) that there exist $\wedge_0$, ρ, $\alpha > 0$ sufciently small such that $I(u) \geq \alpha$ with $\lambda \in (0, \wedge_0)$, $\|u\| = t_1 = \rho$. $\square$

Lemma 2 There exists $\phi \in X$ $\phi \neq 0$ such that $\lim_{t\to+\infty} I(t\phi) \to -\infty$.
Proof Let $\phi \in C_0^\infty(\Omega_0), \phi \geq 0$ and there exist $x \in \Omega_0^c = \Omega \backslash \overline{\Omega_0^c}$ such that for any $x \in \Omega_0^c$ we have $\phi(x) = 0$

$$I(t\phi) = \frac{at^p}{p}\|\phi\|^p + \frac{bt^{p(k+1)}}{p(k+1)}\|\phi\|^{p(k+1)} - \frac{\lambda t^q}{q}\int_\Omega f(x)|\phi|^q dx - -\frac{t^r}{r}\int_\Omega g(x)|\phi|^r dx$$

where $\Omega_0 \subset \Omega_2$ is a bounded domain. Since $q < p(k+1) < r$,we have $I(t\phi) \to -\infty$ as $t \to +\infty$. Therefore, there existst $t_1 \in X$ large enough, such that $I(t\phi) < 0$. $\square$

By Lemmas 1 and 2 and the mountain pass theorem of Ambrosetti and Rabinowitz [8], we deduce the existence of a sequence (u_n) such that

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ in } X^* \tag{11}$$

We prove that (u_n) is bounded in X. Assume for the sake of contradiction, if necessary to a subsequence, still denote by (u_n), $\|(u_n)\| \to \infty$ and $\|(u_n)\| > 1$ for all n.

For n large enough, we have

$$c + 1 + \|u_n\| \geq I(u_n) - \frac{1}{r}\langle I'(u_n), u_n\rangle$$
$$= (\frac{1}{p} - \frac{1}{r})a\|(u_n)\|^p + (\frac{1}{p(k+1)} - \frac{1}{r})b\|(u_n)\|^{p(k+1)} + \lambda(\frac{1}{r} - \frac{1}{q})C_1\|(u_n)\|^q$$
$$c + 1 + \|u_n\| \geq \frac{ar - ap}{pr}\|(u_n)\|^p. \tag{12}$$

Dividing the above inequality by $\|(u_n)\|^p$ and passing to the limit as $n \to \infty$ we obtain a contradiction. It follows that by $\|(u_n)\|$ is bounded in X. This information implies that there exists a subsequence, still denoted by (u_n), and $u_1 \in X$ such that $u_n \rightharpoonup u_1$ in X.

Let B_ρ is an open ball centered at the origin and of radius ρ. Since $f(x) \in L^{q_0}(\Omega)$, for every $\epsilon > 0$, there exists $\rho_0 > 0$ such that

$$\int_{\Omega_\rho^c} |f(x)|^{q_0} < \epsilon, \text{ for} \rho \geq \rho_0 \tag{13}$$

where Ω_ρ is the complement of a smooth bounded domain D in open ball B_ρ and $\Omega_\rho^c = \Omega\backslash\Omega_\rho$. Let ρ be so large that $D \subset B_\rho$. Using Sobolev compact embedding theorem in the bounded domain Ω_ρ, (u_n) has a subsequence, still denoted by (u_n), and u_1 such that $u_n \to u_1$ in $L^q(\Omega_\rho)$.

Since $f(x)$ is bounded in Ω_ρ and (u_n) is bounded in $L^{p^*}(\Omega)$ then by Holder′s inequality, we have

$$\int_\Omega |f(x)||u_n - u_1|^q dx \leq \|f(x)\|_\infty \int_{\Omega\rho} |u_n - u_1|^q dx + \left(\int_{\Omega_\rho^c} |f(x)|^{q_0} dx\right)^{1/q_0} \left(\int_{\Omega_\rho^c} |u_n - u_1|^{p^*} dx\right)^{q/p^*}$$
$$\to 0 \text{ as } n \to \infty \tag{14}$$

So we deduce that using Holder′s inequality

$$\int_\Omega |f(x)||u_n|^{q-2}|u_n||u_n-u_1|dx \leq \left(\int_\Omega |f||u_n-u_1|^q dx\right)^{1/q}\left(\int_\Omega |h||u_n|^q\right)^{(q-1)/q} \to 0 \text{ as } n\to\infty. \tag{15}$$

Similarly, we can derive that

$$\int_\Omega |g(x)||u_n|^{r-2}|u_n||u_n-u_1|dx \to 0 \text{ as } n\to\infty \tag{16}$$

Also using the fact $I'(u_n)\to 0$ in X^* implies that

$$\begin{aligned}\langle I'(u_n), u_n-u_1\rangle &= \left(a+b\|u_n\|^{pk}\right)\int_\Omega |\Delta u_n|^{p-2}\Delta u_n\Delta(u_n-u_1)dx - \lambda\int_\Omega f(x)|u_n|^{q-2}u_n(u_n-u_1)dx\\ &- \int_\Omega g(x)|u_n|^{r-2}u_n(u_n-u_1)dx \to 0 \end{aligned} \tag{17}$$

Then it follows from (15) and (16)-(17) that

$$\left(a+b\|u_n\|^{pk}\right)\int_\Omega |\Delta u_n|^{p-2}\Delta u_n\Delta(u_n-u_1)dx \to 0 \text{ as } n\to\infty \tag{18}$$

Futhermore, the fact $u_n \rightharpoonup u_1$ in X implies

$$\left(a+b\|u_n\|^{pk}\right)\int_\Omega |\Delta u_1|^{p-2}\Delta u_1\Delta(u_n-u_1)dx \to 0 \text{ as } n\to\infty \tag{19}$$

From (18-19), we have

$$\left(a+b\|u_n\|^{pk}\right)\int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u_1|^{p-2}\Delta u_1\right)\Delta(u_n-u_1)dx \to 0 \text{ as } n\to\infty \tag{20}$$

So we deduce that $u_n\to u_1$ in X. Then by relation (11) we have

$$I(u_1)=c\geq 0 \text{ and } I'(u_1)=0, \tag{21}$$

that is, u_1 is a non-trivial weak solution of (1). Now we seek for the solution u_2.

Leema 3 There exists $\phi_1\in X, \phi_1\neq 0$ such that $I(t\phi_1)<0$ for $t>0$ small enough.
Proof Let $\phi_1\in C_0^\infty(\Omega)$, then for small $t>0$

$$I(t\phi_1) = \frac{at^p}{p}\|\phi_1\|^p + \frac{bt^{p(k+1)}}{p(k+1)}\|\phi_1\|^{p(k+1)} - \frac{\lambda t^q}{q}\int_\Omega f(x)|\phi_1|^q dx - -\frac{t^r}{r}\int_\Omega g(x)|\phi_1|^r dx < 0. \quad \square$$

By Lemma 1 it follows that on the boundary of the ball centered at the origin and of radius ρ in X, we have

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$$\inf_{u\in\partial B_\rho} \mathrm{I} > 0 \tag{22}$$

Moreover, for $u \in B_\rho \subset X$, we have

$$-\infty < c_1 = \inf_{\overline{B_\rho}} \mathrm{I} < 0$$

We let now $0 < \epsilon < \inf_{u\in\partial B_\rho} \mathrm{I} - \inf_{u\in B_\rho} \mathrm{I}$ as $\epsilon \downarrow 0$. Applying Ekeland's variational principle [7] to the functional $I\colon \overline{B_\rho} \to \mathbb{R}$, we find $u \in \overline{B_\rho}$ such that

$$I(u_\epsilon) < c_1 + \epsilon \quad \text{and} \quad I(u_\epsilon) < I(u) + \epsilon\|u - u_\epsilon\|, \qquad u \neq u_\epsilon \tag{23}$$

Then it follows that

$$I(u_\epsilon) < c_1 + \epsilon \leq \inf_{B_\rho} \mathrm{I} + \epsilon < \inf_{\partial B_\rho} \mathrm{I},$$

we deduce that $u_\epsilon \in B_\rho$. Now, we define the functional $K\colon B_\rho \to \mathbb{R}$ by $K(u) = I(u) + \epsilon\|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of K. Moreover

$$\frac{K(u_\epsilon + tv) - K(u_\epsilon)}{t} \geq 0, \;\; \text{for small } t > 0 \text{ and } \forall v \in B_{\rho_1} \tag{24}$$

Hence,

$$\frac{I(u_\epsilon + tv) - I(u_\epsilon)}{t} + \epsilon\|v\| \geq 0, \tag{25}$$

Letting $t \to 0$ it follows that

$$\langle I'(u_\epsilon), v\rangle + \epsilon\|v\| \geq 0 \qquad \forall v \in B_\rho. \tag{26}$$

Replacing v in (26) by $-v$, we get

$$-\langle I'(u_\epsilon), v\rangle + \epsilon\|v\| \geq 0 \qquad \forall v \in B_\rho. \tag{27}$$

So that $\|I'(u_\epsilon)\| \leq \epsilon$. We deduce that there exists a sequence $(v_n) \subset B_\rho$ such that

$$I(v_n) \to c_1 \text{ and } I'(v_n) \to 0. \tag{28}$$

It is clear that (v_n) is bounded in X. Thus, there exists $u_2 \in X$ such that, up to a subsequence, (v_n) converges weakly to u_2 in X. Actually, with similar arguments as those used in the proof that the sequence $u_n \to u_1$ in X we can show that $v_n \to u_2$ in X. Thus, by relation (28),

$$I(u_2) = c_1 < 0 \text{ and } I'(u_2) = 0, \tag{29}$$

i.e., u_2 is a non-trivial weak solution of (1). Finally, since

$$I(u_1) = c > 0 > c_1 = I(u_2), \tag{30}$$

we see that $u_1 \neq u_2$. Thus, problem (1) has two non-trivial weak solutions. Then the proof of Theorem 1 is complete. □

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