Fuzzy Sets in KM-algebras

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Abstract

In this paper our aim is to introduce and study fuzzy KM-subalgebras and explore some of its properties. The conception of level KM-subalgebra, strong level KM-subalgebra, super strong level KM-subalgebra are instituted from some fuzzy sets and its descriptions are given.

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1 Introduction

Among so many algebraic structure, one of the important form of class of algebras is the algebra of logic.Some of these examples are BCK-algebras,BCI-algebras,BCH-algebras,KU-algebras,SU-algebras,BF-algebras,UP-algebras and so on.These are connected strongly with logic.One such example,BCI-algebras and BCK-algebras introduced by Iseki and Imai are two classes of logical algebras. Many researchers have been extensively investigating these two logical algebras introduced by Iseki and Imai.

Fuzzy set theory was first introduced by Zadeh in 1965 and Goguen in 1967 show the intention of the authors to generalize the classical notion of a set and a proposition to accommodate fuzziness. There have been a wide range of applications of the fuzzy set theories in the domain of mathematics and elsewhere.

Jun in 2000 introduced a new notion of M-BCK/BCI-algebras and analysed some their properties.Also Jun investigated Q-fuzzy subalgebras of BCK/BCI algebras and contributed some examples.Ahn and Bang in 2003 introduced the concept of level subalgebras in B-algebras.Akram and Dar in 2005 introduced the concept of T-fuzzy subalgebras. Saeid and Jun in 2008 ,by using the notion of anti fuzzy points introduced the concept of anti fuzzy subalgebras of BCK/BCI algebras.In 2009,fuzzy BF-algebras and fuzzy topological BF-algebras was introduced by Saeid and Rezvani.In 2015,Jaruwat Somjanta,NatthanichaThuekaew,Praprisri Kumpeangkeaw,Aiyared Iampan introduced the notion of Fuzzy sets in UP-algebra.In 2019,S.Sowmiya and P.Jeyalakshmi introduced the notion of Fuzzy Algebraic Structure in Z-algebras.

Now we introduced a new concept of KM-algebras.fuzzy subalgebra plays a vital role in studying many logical algebras.In this paper,we introduced the notions of fuzzy KM-subalgebras,level ,strong level,super strong level KM-subalgebras and their properties are investigated.

2 KM-Algebras and its Properties

2.1 Definition

A ***KM-algebra*** is an algebra $(X, *, 0)$of type $(2, 0)$ satisfying the following conditions:

i) $x * x = 0$
ii) $x * 0 = 0$
iii) $(x * y) * z = (x * z) * y$
iv) $x * y = 0 \ and \ y * x = 0 \ \Rightarrow \ x = y, \ for \ all \ x, y, z \in X.$

2.2 Definition

Let S be a non-empty subset of a KM-algebra X, then S is called a ***KM-subalgebra*** of X if $x * y \in S, \ for \ all \ x, y \in S.$

Example

Let $X = \{0, 1, 2\}$ be a set with the following Cayley table

*	0	1	2
0	0	1	2
1	0	0	2
2	0	2	0

Then $(X, *, 0)$ is a KM-algebra.

2.3 Proposition

In any KM-algebra $(X, *, 0)$ with $x \leq y$, the following holds good for all $x, y \in X$.

i) $x * (y * x) = y * (x * x)$
ii) $y * (x * (y * x)) = 0$
iii) $(x * (x * y)) * y = (x * x) * (y * y)$
iv) $y * (y * (y * x)) = y * x$
v) $x * (x * y) = x * y$
vi) $y * (x * (x * y)) = 0$
vii) $(x * y) * 0 = (x * 0) * (y * 0)$
viii) $(x * y) * x = 0 * y$.

3 Main Results

3.1 Definition

Let μ be a fuzzy set in a KM-algebra. Then μ is called a fuzzy KM-subalgebra of X if $\mu(x * y) \geq min \mu(x), \mu(y)$ for all x,y ϵX .

3.2 Example

Let X={0,1,2} be a set with the following table

*	0	1	2
0	0	1	2
1	0	0	2
2	0	2	0

$(X, *, 0)$ is a KM-algebra.
Define a fuzzy set $\mu : X \longrightarrow [0, 1]$ by $\mu(0) = 0.5$,$\mu(1) = 0.3$,$\mu(2) = 0.1$.
Then μ is a fuzzy KM-subalgebra of X.

If $A \subseteq X$, the characteristic function $\mu_A$of X is a function of X into$\{0, 1\}$defined as

$$\mu_A(X) = \begin{cases} 1\, if \;\; x \in A \\ 0\, if \;\; x \notin A \end{cases}$$

3.3 Example

Let $X = \{0, 1, 2\}$be a KM-algebra of $A = \{0, 1, 2, 3\}$ .Then $\mu(x) = \begin{cases} 1\ if\ x \in \{1, 2, 3\} \\ 0\ if\ x \in \{0\} \end{cases}$ is a fuzzy KM-subalgebras of A.

3.4 Lemma

If A is a fuzzy KM-subalgebra of X, then for all $x \in X\ \ \mu_A(x)$.

Proof:

For all $x \in X$, we have $x * x = 0$.
Hence $\mu_A(0) = \mu_A(x * x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$

$$\mu_A(0) \geq \mu_A(x)$$

3.5 Theorem

Let B be a non empty subset of A. Then B is a KM-subalgebra of A if the characteristic function
μ_B is a fuzzy KM-subalgebra of A.

Proof:

Assume that B is a KM-subalgebra of A.

Case (i):

suppose $x, y \in B$then $\mu_B(x) = 1\ and\ \mu_B(y) = 1$.

Therefore $\min\{\mu_B(x), \mu_B(y)\} = \min\{1, 1\} = 1$.

Since B is a KM-subalgebra of A, we have $x \times y \in B$

$$\Rightarrow \mu_B(x * y) = 1$$
$$\Rightarrow \mu_B(x * y) = 1 \geq 1 = \min\{\mu_B(x), \mu_B(y)\}$$

Case (ii):

Suppose $x \notin B\, y \notin B$. Then $\mu_B(x)$=0 or $\mu_B(y) = 0$.

Thus $\min\{\mu_B(x), \mu_B(y)\} = 0$..
Therefore $\mu_B(x * y) \geq 0 = \min\{\mu_B(x), \mu_B(y)\}$.
Hence $\mu_B$is a fuzzy KM-subalgebra of A.

Conversely, assume that $\mu_B$is a fuzzy KM-subalgebra of A.
Let $x, y \in B$ then $\mu_B(x) = 1$ & $\mu_B(y) = 1$.

$$\Rightarrow \min\{\mu_B(x), \mu_B(y)\} = 1$$

Since $\mu_B$is a fuzzy KM-subalgebra of A,
we have $\mu_B(x * y) \geq \min\{f_B(x), f_B(y)\} = 1$

$$\Rightarrow \mu_B(x * y) = 1$$
$$\Rightarrow x * y \in B$$

B is a KM-subalgebra of A.

3.6 Proposition

Let A be a fuzzy KM-subalgebra of X, & let $n \in N$. Then

1. $\mu_A(\prod^n x * x) \geq \mu_A(x)$,for any odd number n
2. $\mu_A(\prod^n x * x) = \mu_A(x), for\ any\ \ even\ Number\ n$

Proof:

1. Let $x \in X$and assume that n is odd.

That is $n = 2k - 1$,for some positive integer k, apply mathematical induction on n.

$$\mu_A(x * x) = \mu_A(0) \geq \mu_A(x)$$

Assume that the result is true for 2k-1
Therefore $\mu_A(\prod^{2k-1} x * x) \geq \mu_A(x)$

$$\begin{aligned}\mu_A(\prod^{(2k+1)} x * x) &= \mu_A(\prod^{2k-1} x * (x * (x * x))) \\ &= \mu_A(\prod^{2k-1} x * 0) \\ &= \mu_A(\prod^{2k-1} x * x) \\ &\geq \mu_A(x)\end{aligned}$$

Which proves (i).

1. Let $x \in X$and assume that n is even.

Let $n = 2k$for some positive integer k. Apply mathematical induction on n

$$\mu_A((x \times x) \times (x \times x)) \geq \min\{\mu_A(x \times x), \mu_A(x \times x)\}$$
$$\geq \min\{\min\{\mu_A(x), \mu_A(x)\}, \min\{\mu_A(x), \mu_A(x)\}$$
$$= \mu_A(x)$$

Which proves (ii).

3.7 Theorem

Let A be a fuzzy KM-subalgebra of X. If there exists a sequence $\{x_n\}$in X, such that
$\lim_{n\to\infty} \mu_{\mathrm{A}}(x_n) = 1$, then $\mu_A(0) = 1$.

Proof:

By above lemma we have $\mu_A(0) \geq \mu_{\mathrm{A}}(x_n)$for every positive integer n.
Consider $1 \geq \mu_{\mathrm{A}}(0) \geq \lim_{n\to\infty} \mu_{\mathrm{A}}(x_n) = 1$.
Hence $\mu_{\mathrm{A}}(0) = 1$.

3.8 Theorem

Let A_1 and A_2 are fuzzy KM-subalgebra of X. Then $A_1 \bigcap A_2$is a fuzzy KM-subalgebra of X.

Proof:

Let $x \in A_1 \bigcap A_2$then $x \in A_1$and $x \in A_2$

$$\mu_{A_1 \bigcap A2}(x \times y) = \min\{\mu_{A_1}(x \times y), \mu_{A_2}(x \times y)\}$$
$$\geq \min\{\min\{\mu_{A_1}(x), \mu_{A_1}(y)\}, \min\{\mu_{A_2}(x), \mu_{A_2}(y)\}\}$$
$$= \min\left\{\mu_{A_1 \bigcap A_2}(x), \mu_{A_1 \bigcap A_2}(y)\right\}$$

Therefore $A_1 \bigcap A_2$ is a fuzzy KM-subalgebra of X.

3.9 Theorem

Let $\{A_i|i \in N\}$be a family of fuzzy KM-subalgebra of X, then $\bigcap\limits_{i\in N} A_i$is also on fuzzy KM-subalgebra of X.

3.10 Definition

Let A be a fuzzy set in KM-algebra of X & $\lambda \in [0,1]$, then the level KM-subalgebra $M(A,\lambda)$of A is defined as
$M(A,\lambda) = \{x * y \in X|\mu_A(x,y) = \lambda\}$(for some $x \in X$).

Given a fuzzy set A defined on KM-algebra X and any number$\lambda \in [0,1]$, then the strong level and the super strong level KM-subalgebra $M(A,>,\lambda)$of X are defined as follows

$$M(A;\geq,\lambda) = \{x * y \in X|\mu_A(x,y) \geq \lambda\}$$

$$M(A;>,\lambda) = \{x * y \in X|\mu_A(x,y) > \lambda\}$$

.

Example
Let $X = \{0,1,2\}$be a set with the following table,

*	0	1	2
0	0	1	2
1	0	0	2
2	0	2	0

Define a fuzzy set $\mu : X \to [0,1]$by $\mu(0) = 0.7$,$\mu(1) = 0.1$,$\mu(2) = 0.3$then μ is a KM-algebra of X.

Level KM-subalgebras,$\mu(A,\lambda) = \{0 * 1, 2 * 2\}$

Strong level KM-algebras,$\mu(A,\geq,\lambda) = \{0 * 0, 0 * 2, 1 * 0, 1 * 1, 1 * 2, 2 * 0, 2 * 1,\}$

3.11 Theorem

A strong level KM-subalgwbra A on R is convex if and only if$\mu(\lambda x_1 + (1 - \lambda x_2)) \geq \min[\mu(x_1), \mu(x_2)]$ for all $x_1, x_2 \in R$and all $\lambda \in [0,1]$, where min denotes the minimum operator.

Proof:

Assume that A is Convex. For any $\alpha \in [0,1]$

$$\alpha = \mu(x_1) \geq \mu(x_2)$$

$$\Rightarrow x_1, x_2 \in M(A; \geq, \alpha)$$

$$\Rightarrow \lambda x_1 + (1 - \lambda x_2) \in M(A; \geq, \alpha) \, \forall \alpha \in [0,1]$$

By the convexity of A.

$$\mu(\lambda x_1 + (1-\lambda) x_2) \geq \alpha = \mu(x_1) = \min[\mu(x_1), \mu(x_2)]$$

$$\Rightarrow \mu(\lambda x_1 + (1-\lambda) x_2) \geq \min[\mu(x_1), \mu(x2)]$$

.
Conversely

Assume that A satisfies

$$\mu(\lambda x_1 + (1-\lambda) x_2) \geq \min[\mu(x_1), \mu(x2)]$$

To prove for any $\alpha \in [0,1], M(A; \geq; \alpha)$is convex.
For any $x_1, x_2 \in M(A; \geq, \alpha)$
Ie, $\mu(x_1) \geq \alpha$ &$\mu(x_2) \geq \alpha$
For any $\lambda \in [0,1]$,

$$\mu(\lambda x_1 + (1-\lambda) x_2) \geq \min[\mu(x_1), \mu(x_2)]$$

$$\geq \min[\alpha, \alpha]$$

$$= \alpha$$

i.e , $(\lambda x_1 + (1-\lambda) x_2) \in M(A; \geq, \alpha) \Rightarrow M(A; \geq, \alpha)$ is convex for any $\alpha \in [0,1]$
Hence A is convex.

3.12 Theorem

Every KM-subalgebra of a strong level KM-subalgebra is a fuzzy KM-subalgebra of X.

Proof:

Let Y be a KM-subalgebra of X and let A be a fuzzy set on X defined by
$\mu_A(x) = \begin{cases} \lambda & if\, x \in Y \\ 0 & otherwise \end{cases}$

Also $M(A, \geq, \lambda) = Y$.

Let $x, y \in X$. We consider the following cases

Case (i):- If $x, y \in X$,then $x * y \in Y$
therefore$\mu_A(x * y) = \lambda = \min\{\lambda, \lambda\} = \min\{\mu_A(x), \mu_A(y)\}$.

Case (ii):- If $x, y \notin Y$,then $x * y \in Y$,then $\mu_A(x) = 0$& $\mu_A(y) = 0$
therefore$\mu_A(x * y) \geq 0 = \min\{0, 0\} = \min\{\mu_A(x), \mu_A(y)\}$.

Case (iii):- If $x \in y$ & $y \notin Y$,then $\mu_A(x) = \lambda$and $\mu_A(y) = 0$
therefore$\mu_A(x * y) \geq 0 = \min\{\lambda, 0\} = \min\{\mu_A(x), \mu_A(y)\}$.

Case(iv):- If $x \notin y$&$y \in Y$,then by the same argument as in case (iii), we can conclude that

$$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

Therefore A is a fuzzy KM-subalgebra of A.

In the next theorem we generalize the above theorem.

3.13 Theorem

Let X be a KM-algebra then for any chain of KM-subalgebras
$A_0 \subset A_1 \subset ... \subset A_r = X$there exists a fuzzy KM-subalgebra on x whose strong level KM-subalgebras are exactly the KM-subalgebras of this chain.

Proof:

Consider a set of numbers
$n_0 > n_1 > ..... > n_r$where each $n_i \in [0, 1]$ define $\mu : X \to [0, 1]$by
$\mu(A_i \backslash A_{i-1}) = t_i$for all$0 < i \leq r$, & $\mu(A_0) = t_0$
To prove that is a fuzzy KM-subalgebra of X.
Let $x, y \in Y$, we consider the following cases

Case (i):

Let $x, y \in A_i \backslash A_{i-1}$, then $\mu(x) = t_i = \mu(y)$
Since $A_i$is a KM-subalgebra, then $x * y \in A_i$
So $x * y \in A_i \backslash A_{i-1}$ or $x * y \in A_{i-1}$ and in each of them we have

$$\mu(x * y) \geq t_i = \min\{\mu(x), \mu(y)\}$$

Case (ii):

Let $x \in A_i \backslash A_{i-1}$, $y \in A_j \backslash A_{j-1}$ where $i < j$then
$\mu(x) = t_i$,$\mu(y) = t_j$since $A_j \subseteq A_i$ & $A_i$is a KM-subalgebra of X, then $x * y \in A_i$
Hence $\mu(x * y) \geq t_j = \min\{\mu(x), \mu(y)\}$
Also $I_m(\mu) = \{t_0, t_1, ...t_r\}$, therefore the strong level KM-subalgebras of $\mu$are given by the chain of KM-subalgebras

$$\mu_{t_0} \subset \mu_{t_1} \subset .... \subset \mu_{t_r} = X$$

We have $\mu_{t0} = \{x \in X | \mu(x) \geq t_0\} = A_0$

$$\Rightarrow A_i \subseteq \mu_{t_i}$$

Let $x \in \mu_{t_i}$then $\mu(x) \geq t_i$then $x \in A_j \ for \ \ j > i$
So $\mu(x) \in \{t_0, t_1...t_i\}$thus $x \in A_k fork \leq i, sinceA_k \subseteq A_i, wegetx \in A_i$
Hence $A_i = \mu_{t_i} \ \ for \ \ 0 \leq i \leq r$
Hence is a fuzzy KM-subalgebra of X.

3.14 Theorem

Let Y be a subset of X & A be a fuzzy set on X which is given in the proof of theorem. If A is a fuzzy KM-subalgebra of X, then Y is a KM-subalgebra of X.

Proof:

Let A be a fuzzy KM-subalgebra of X, & $x, y \in Y$
Then $\mu_A(x) = \mu_A(y) = \lambda$
Therefore $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$

$$= \min\{\lambda, \lambda\}$$

$$= \lambda$$

$$\Rightarrow x * y \in Y$$

$\Rightarrow Y$is a KM-subalgebra of X.

3.15 Theorem

If A is a fuzzy KM-subalgebra of X,then the set$X_{\mu A} = \{x \in X | \mu_A(x) = \mu_A(0)\}$is a KM-subalgebra of X.

Proof:

Let $x, y \in X_{\mu_A}$
Then $\mu_A(x) = \mu_A(0) = \mu_A(y)$

$$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

$$= \min\{\mu_A(0), \mu_A(0)\}$$

$$= \mu_A(0)$$

By lemma $\mu_A(x * y) = \mu_A(0) \Rightarrow x * y \in X_{\mu_A}$
$\Rightarrow X_{\mu_A}$is a fuzzy KM-subalgebra of X.

3.16 Theorem

Let M be a crisp subset of X. suppose that N is a fuzzy set of X defined by$\mu_N$as

$$\mu_N(x) = \begin{cases} \alpha \;\; if \; x \in M \\ \beta \;\; otherwise \end{cases}$$

For all $\alpha, \beta \in [0, 1]$with $\alpha \geq \beta$. Then N is a fuzzy KM-subalgebra iff M is a(crisp) KM-subalgebra of X.

Proof:

Let N be a fuzzy KM-subalgebra
Let $x, y \in X$ be such that$x, y \in M$
Then $\mu_N(x * y) \geq \min\{\mu_N(x), \mu_N(y)\}$

$$= \min\{\alpha, \alpha\}$$

$$= \alpha$$

$$\Rightarrow x * y \in \mu$$

$\Rightarrow M$is a KM-Subalgebra of X. conversely, suppose that M is a KM-Subalgebra of X.

Let $x, y \in X$.

1. If $x, y \in M$ then$x * y \in M$,thus

$$\mu_N (x * y) = \alpha = \min \{\mu_N (x), \mu_N (y)\}$$

2. If $x \notin M\ (or)\ y \notin M$,then

$$\mu_N (x * y) \geq \beta = \min \{\mu_N (x), \mu_N (y)\}$$

$\Rightarrow$N is a fuzzy KM-subalgebra , moreover we have

$$X_{\mu_N} = \{x \in X | \mu_N (x) = \mu_N (0)\}$$

$$= \{x \in X | \mu_N (x) = \alpha\}$$

X_{μ_N}=M.

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