Appendage of inverse graphs of a group

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Abstract:
In this paper, The Appendage graph of the inverse graph of a cyclic group was studied and then investigated the dimension of Appendage graph and different properties on the order of the graph. Later we discussed about the chromatic number of Appendage graph and establish a relation between the maximum degree of the graph and the chromatic number.
Key words:
Cyclic group, Inverse graph, Appendage graph, tripartite graph, chromatic number.

Introduction:

Groups and graphs have been related from long time. It is very easy to understand for a reader if we present the typical group axioms as graphs. In this paper we introduced the Appendage of the inverse graph, its dimension, chromatic number and its properties. In our previous work we discussed about the inverse graph and its important algebraic properties. In [13] and [14], N. Naga Maruthi Kumari has worked on Isomorhism of groups and graphs and also worked on relation between the cyclic groups and generator graphs.

Fundamental ideas:

Definition 1: An algebraic structure G with binary operation " $\circ$" denoted by (G, $\circ$) is said to be a group, if it satisfies the following 4 properties:
1. Closure property: $for\ all\ x, y \in G, x \circ y \in G$
2. Associative property: for all , $y\ and\ z \in G$, $x\circ (y \circ z) = (x \circ y) \circ z$
3. Identity property: $for\ every\ x \in G, \exists\ e \in \ni x \circ e = e \circ x = a$, then 'e' is called the identity element under the binary operation $\circ$
4. Inverse property: $for\ every\ x \in G, \exists\ x -1 \in G \ni x \circ x\ 1$ = e, then $x -1$ is called
of x and vice versa.[1]

Definition 2: A group G is called a cyclic group if ∃ an element $x \in G \ni$ every element of G can be expressed as power of x. In this case x is called generator of G. This can be written as G=< x> [2]

Definition 3: A graph G is an ordered pair G= (V, E) where V is a nonempty set of elements which are called as vertices or nodes and E is the set of elements which are connecting the elements of V, are called as edges of the graph is G. [3]

Definition 4: Let $(\tau, \circ)$ be a finite group and S = $\{ v \in \tau \mid v \neq v^{-1}\}$. In this paper S always denotes the non-self-invertible elements of the group τ. [4]

Definition 5: The chromatic number of a graph is the minimum number of colors needed to properly color the vertices of the graph so that no two adjacent vertices share the same color. It is denoted by (G) [5]

Definition 6: A Bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to a vertex in V and no two vertices are adjacent in the same set [6]

Definition 7: A tripartite graph is a set of graph vertices decomposed into three disjoint sets such that no two vertices with in the same set are adjacent [7]

Procedure to draw the Appendage graph (Mycielski's construction)

Let the vertices of the inverse graph $G_S(\tau)$ be $x_{1,2}, x_3, \ldots\ldots x_n$.

Form a new graph E(τ) from $G_s(\tau, \circ)$as follows:

Add n+1 vertices $x_1', {}_{2}', x_3' \ldots\ldots x_n', u$(key node) and then for $1 \leq i \leq n$, join x_i' to the neighbors of x_i and to the key node u.

For example if $G_S(Z_3)$ is K_3 and extension graph of Gs (Z_3) *is denoted by* $E_S(Z3)$ [8]

Example: consider $(Z4, +)$ is a group. S is a nonempty set of non-self- invertible elements of $Z4$. $Z4$= {0, 1, 2, 3} and S = {1, 3}. The Appendage graph E S ($\boldsymbol{Z}4$) will be represented as follows:

Remark: **Coloring procedure of Appendage graph**: If the key node $'u'$ is colored with (say) color $'a'$ and the remaining n vertices $x_1', {}_{2}', x_3' \ldots\ldots x_n'$of the Appendage graph $E_S(Zn)$ should be colored with the other color (say) $'b'$. The identity node can be colored with the color of key node and the remaining vertices of inverse graph of $G_S(\tau)$ can be properly colored

Theorem 1: If the Chromatic number of the inverse graph of (τ) is k then the chromatic number of the Appendage graph $E(\tau)$ is k+1

Proof:
Let $Gs(\tau, \circ)$ is the inverse graph of the group $(\tau, \circ)$ and S be the set of non self invertible elements of the group$(\tau, \circ)$
Let the Appendage graph of $Gs(\tau, \circ)$ denoted by $Es(\tau, \circ)$
We will prove the result by induction on n.
If n=1 or n=2 the result is trivial, hence will start the induction on n =3

If χ $(Gs(\tau, \circ))$=3, then $\chi(E_S(\tau))$=4 since $Es(\tau)$ is a Tripartite graph with a key node $u \in Es(\tau)$ we have to assign a new color to the key node u
Similarly if χ $(Gs(\tau))$=4 then $\chi(E_S(\tau))$=5 and $u \in Es(\tau)$
Assume that the result is true for (k-1)
i.e., If χ $(Gs(\tau))$=k−1 then $\chi(E_S(\tau))$=k
By induction if χ $(Gs(\tau))$=k then
χ $(Gs(\tau, \circ))$ =$(m/2)$+1 where m=$|S|$ [9]
In the Appendage graph $E_S(\tau)$ as we have a key node $u \in Es(\tau)$ we have to assign a new color other than k colors to the key node u and hence
χ $(E_S(\tau))$=$(|S|/2)$+1+1 for each $v_i \in$ $E_S(\tau)$
χ $(E_S(\tau))$= k+1
Hence the chromatic number of the Appendage graph of $(\tau, \circ)$ is k+1

Theorem 2: In the inverse graph of (τ) the identity $'e'$ of the group $(\tau, \circ)$ is analogous to the key node $u \in Es(\tau)$ if n is odd.
Proof:
Let (τ) is a inverse graph of $(\tau, \circ)$ and S be a set of non self-invertible elements of the group $(\tau, \circ)$
Let (τ) be the Appendage graph of $G(\tau)$
We will prove this by using contradiction
Let $'e'$ be the identity element in the group $(\tau, \circ)$
Let us assume that, n to be even, then $e \in (\tau)$ is not adjacent to $(n/2)th$ node in $Gs(\tau)$
Hence (e)=n−2
In the (τ), the key node $u \in E(\tau)$ is of degree n
But by the Mycielski's construction [10] the key node u must be adjacent to all n nodes
Hence e is not analogous to u

i.e., $(e)\neq d(v)$
Hence n cannot be even
Therefore n must be odd
Remark: Jn the inverse graph $(Zn,\circ)$ the identity node is not adjacent with $(n/2)th$ node and the degree of the identity node is $(n$ -2) if n is even. Hence its Appendage graph $(Zn,\circ)$ the degree of Identity node is $2(n-2)$

Theorem 3: The identity **node** $e\in(\tau)$ is of degree $2(n-2)$ if n is even and $2(n-1)$ if n is odd

Proof:
Case-i: if n is even
In (τ), for $e\in$, e must be adjacent to all the vertices say $x1,x2,x3,\ldots\ldots.xn2-1,xn2+1\ldots..xn-1$
Then e is adjacent with $(n-2)$ vertices i.e. $(n-2)$ elements of τ, except the self-invertible elements
That is (e) in (τ) is $(n-2)$
In (τ), as every vertex x_i' is adjacent with e for i=1,2,3,…….$n/2-1,n/2+1\ldots\ldots n$
Hence (e) in (τ) is $(n-2)+(n-2)$
Therefore is $(e)=2(n-2)$
Case-ii: if n is odd
In (τ), for $e\in$, e must be adjacent to all $(n-1)$ vertices say $x_1,x_2,x_3,\ldots\ldots..x_{n-1}$
I.e., (e) in (τ) is $(n-1)$
In (τ), as every vertex x_i' is adjacent with e for i=1,2,3,…….$(n-1)$
Hence (e) in (τ) is $(n-1)+(n-1)$
Therefore is $(e)=2(n-1)$

Theorem 4: In (τ) the identity element $e\in\tau$ has the maximum degree

Proof: Let $x_i\in\tau$ $(x_i\neq e)$ be a node in (τ) which is of maximum degree
$\Rightarrow(e)<d(xi)$ for some i …… (1)
Case 1 : If n is even
$(e)=2(n-2)$……(2)
We know that each x_i $((i=1,2,3,\ldots\ldots.n)\in Gs(\tau)$ is not adjacent to all $(n-1)$ vertices
$(xi)\leq(n-1)$……..(3)
From (1),(2) and (3) $\Rightarrow d(e)=2n-4<d(xi)\leq(n-1)$
$\Rightarrow 2n-4<(n-1)\Rightarrow n<3$
Which is absurd as $n\in N$
Hence (e) must be maximum
That is $(x_i)<(e)\ \forall\ i$
Case 2 : If n is odd,
$(e)=2(n-1)$……(4)
As each x_i $((i=1,2,3,\ldots\ldots.n)\in Gs(\tau)$ is not adjacent to all $(n-1)$ vertices

$(xi) \leq (n-1)$ …….(5)

From (4) and (5) $\Rightarrow (e)=2n-2<d(xi) \leq (n-1)$
$\Rightarrow 2n-2<(n-1) \Rightarrow n<1$
Which is absurd as $n \in N$
Hence (e) must be maximum
That is $(xi) < (e)\ \forall\ i$

Theorem 5:
The Appendage inverse graph (τ) is a triangle free graph

Proof: Let (τ) is a inverse graph of (τ,∘) and S be a set of non-self-invertible elements of the group (τ,∘)
Let $x1,x2,x3,\ldots\ldots xn$ be the nodes of $Gs(\tau)$, add n+1 vertices $x1',x2',x3'\ldots\ldots xn',u$
In $Es(\tau)$ we find three disjoint sets A=$\{x_1,x_2,x_3,\ldots\ldots x_n\}$,$A'$={ $x1',x2',x3'\ldots\ldots xn'$,} and $U=\{u\}$
Such that $A \cup A' \cup U$=(τ) and $A \cap A' \cap U=\varphi$
∴ (τ) is a tripartite graph and hence it is triangle free graph.

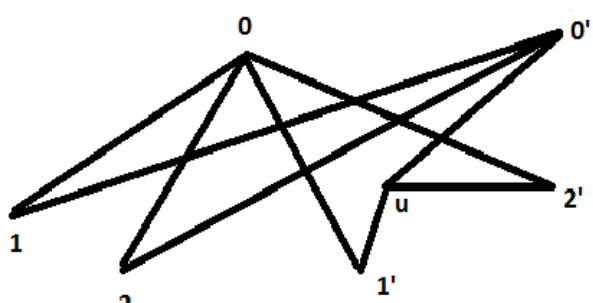

Figure 2

Theorem 6: If ((τ),,E) is the inverse graph of the cyclic group (τ,∘), then the Appendage inverse graph $(Es(\tau),V',E')$ contains $2n+1$ vertices and $3\Sigma d(xi)+2n$ edges

Proof: Let ((τ),,E) is the inverse graph of the cyclic group (τ,∘),
Let S be the set of all non-self-invertible elements of (τ,∘)
By the Mycielski's construction [11] the Appendage graph (τ) contains $2n+1$ vertices,
$\{x1,2,x3,\ldots\ldots xn,x1',x2',x3'\ldots\ldots xn',u\}$ Such that
$(xi')=(xi)+1$ for all i=1,2,3-----n
And $(u)=n$
Also $\forall\ xi \in$ (τ) ,$d(xi)$ becomes double in $Es(\tau)$, therefore
$\Sigma d(v')=2\Sigma d(xi)+\Sigma d(xi')+d(u)$ for $v' \in V'(Es(\tau))$
$=2\Sigma(xi)+\Sigma d(xi)+\Sigma 1+d(u)$
$=2\Sigma(xi)+\Sigma d(xi)+n+n$

$=3\Sigma(xi)+2n$

Therefore $|E|=(3\Sigma d(xi)+2n)/2$(By Hand Shaking Property) [12]

Theorem 7: The Chromatic number of Appendage graph cannot exceed the degree of the identity element of the group $(\tau,\circ)$ i.e.,$\chi(E_s(\tau))\leq\Delta(E_s(\tau))=d(e)$

Proof:

Let (τ) be a Appendage inverse graph and S be a non-self-invertible elements of $(\tau,\circ)$

Let us suppose that $(e)\leq\chi(Es(\tau))$

The identity node e has $(n-1)$ neighbours in (τ) and again $(n-1)$ neighbours in $E(\tau)$, if n is odd

The identity node e has $(n-2)$ neighbours in (τ) and again $(n-2)$ neighbours in $E(\tau)$, if n is even.

Hence (e) is either twice of $(n-1)$ or twice of $((n-2))$

$\Rightarrow(e)$ is always even

By taking the nodes other than $'e'$ in $Gs(\tau)$ as A = $\{x1,x2\ ,x3,\ldots xn-1\}$, B = $\{x1',x2\ ',x3,'\ldots\ldots xn-1'\}$ where$x1',x2\ ',x3,'\ldots\ldots xn-1'\in Es(\tau)$ and $C=\{u\}$ are the disjoint sets and does not have any edge whose end vertices are in the same set.

$\therefore$ The Appendage graph (τ) must be colored with fewer colors than degree of e

This is a contradiction to the assumption that $(e)\leq\chi(Es(\tau))$

Hence $\chi(Es(\tau))\leq\Delta(Es(\tau))$

Conclusion:

In this paper as explained in the abstract we constructed Appendage graph by using Mycielski's construction. Later we proved the properties of Appendage graph, its chromatic number and established a relation between its maximum degree and chromatic number. We want to extend this study to find chromatic polynomial of inverse graph and its Appendage graph. Later we want to continue this study in the advanced applications of networking and coding

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