A Novel Approximate Analytical approach to solve the Boussinesq equation

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Abstract:

The water table level is predicted via unconfined aquifer interacting with stream, which is modelled by the non-linear Boussinesq equation. The present paper aims to find an Approximate Analytical solution by Variational Homotopy Perturbation Method of nonlinear Boussinesq equation. The comparison with VHPM solution with exact solution emphasizes the high accuracy of the method. The Variational Homotopy Perturbation Solution presented in the paper show remarkable high precision and its ease of application to nonlinear problems

Keywords: Boussinesq equation, analytical method, hydrology.

I. INTRODUCTION

The ever increasing demand of water leads to lack of water resources at various places in the world on account of population hike and the fast development of industry and agriculture. Due to this reason various strategies were proposed by engineers and planners for water resources system. There is a requirement for developing competent mathematical model that can illustrate both spatial and temporal allocation of the groundwater head and its zones connected with water sources.

The interaction of the aquifer and the river was developed in the earliest study by Theis(1941).Most of the saturated flow modeling studies carried out in India has been related to unconfined alluvial aquifer bounded by rivers. The Dupuit assumptions are the most powerful tool for engineers and hydrologists for treating unconfined flows.Several developments in connection with the stream aquifer problems have been considered from different viewpoints. A variety of solutions of the Boussinesq equation are studied

.Minute variation in the height of the stream can be a source a great change in the height of groundwater of the aquifer. Parlange et. al. (2000) [6] originated an exact solution of the Boussinesq equation which describes water movement, he has also discussed the case of finite aquifer. Srivastava (2003) [8] analyzed the response of an aquifer to a stream with stage linearly increasing with Time. Workmann et al. (2008) [7] studied the stream-aquifer interaction with the help of sand tank model together with mathematical modeling by the Boussinesq equation. Ganji et. al (2011)[3] obtained solution of Boussinesq equation based on homotopy perturbation method. They considered one side time varying boundary condition whereas another side no flow condition and obtained analytic solutions compared with the exact solution.

In the present paper the stream–aquifer system is studied with the help of mathematical modeling by means of the Boussinesq equation when the water head at the source is a random function of time. The non linear Boussinesq equation is solved by Variational Homotopy Perturbation Method [2,9] which is a graceful blend of Variational Iteration Method and Homotopy Perturbation Method.

II. PROBLEM FORMULATION

The principal equation for one-dimensional unconfined groundwater flow by assuming one dimensional horizontal ground water flow in a homogeneous and isotropic aquifer shown by Fig. 1 is the Boussinesq equation,

$$\frac{\partial h}{\partial t}=\frac{K}{S_y}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) \tag{1}$$

where the hydraulic head (m) is represented by h(x,t); K defines the hydraulic conductivity of the aquifer (m/day); S being the aquifer specific yield.

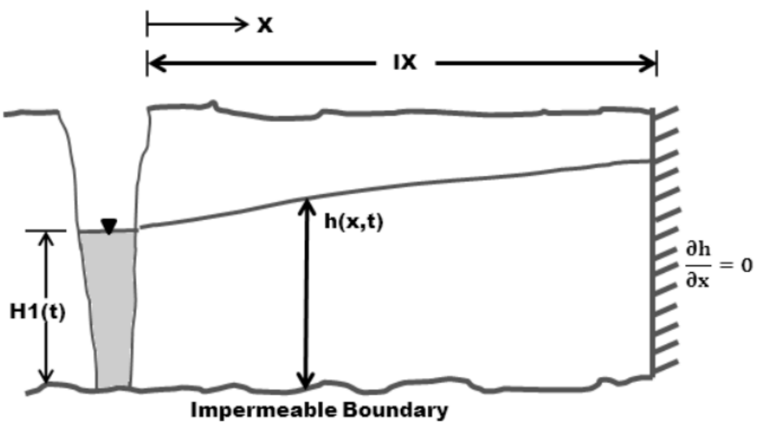

Fig. 1. Schematic diagram of stream-unconfined aquifer interaction

The boundary conditions are given by

$$h(0,t)=H(t) \tag{2}$$

$$\frac{\partial h(l_x,t)}{\partial x}=0\ , \qquad t\geq 0 \tag{3}$$

where represents the water height at a distance *x* from the origin and time t
the exact solution of equation(1) – (3) is

$$h(x,t)=\frac{x}{t+1}-\frac{x^2}{6(t+1)}+\frac{3}{2(t+1)}\left[(t+1)^{2/3}-1\right] \tag{4}$$

In the above boundary condition we consider the case where $H(t)$ increases from zero to a maximum and then decreases back to zero, thus providing a realistic behaviour for all times.

In the boundary condition(2) zero flux is considered at the impervious surface $x = l_x$.

The initial condition is expressed by

$$h(x,0)=H_0(x) \tag{5}$$

We suppose initial water head in the aquifer as a quadratic approximation and choose $H_0(x)=a\,x^2+b\,x+c$ .Under the above initial and boundary conditions, the solution of Boussinesq equation (1) is obtained which is shown in the next section.

III. Approximate analytical solution of Boussinesq equation for horizontal aquifer

According to Variational Homotopy Perturbation Method, correction functional is constructed for equation (1) as

$$h_{n+1}(x,t)=h_n(x,t)+\int_0^t\left[\lambda(\tau)\left(\frac{\partial h_n}{\partial\tau}-\frac{K}{S_y}\left(\frac{\partial h_n}{\partial x}\right)^2-\frac{K}{S_y}h_n\left(\frac{\partial^2 h_n}{\partial x^2}\right)\right)\right]d\tau$$

which yields the stationary conditions

$$1+\lambda(\tau)=0$$

$$\frac{\partial}{\partial\tau}\lambda(\tau)=0$$

Therefore, the general Lagrange multiplier can be readily identified as $\lambda=-1$, which yields the following iteration formula

$$h_{n+1}(x,t)=h_n(x,t)-\int_0^t\left[\left(\frac{\partial h_n}{\partial\tau}-\frac{K}{S_y}\left(\frac{\partial h_n}{\partial x}\right)^2-\frac{K}{S_y}h_n\left(\frac{\partial^2 h_n}{\partial x^2}\right)\right)\right]d\tau$$

Applying the variational homotopy perturbation method, we get

$$\sum_{n=0}^{\infty}p^n\left(h_n(x,t)\right)=\left(ax^2+bx+c\right)+p\frac{K}{S_y}\int_0^t\left[\begin{array}{l}\left(\sum_{n=0}^{\infty}p^n\left(h_n(x,t)\right)\right)\left(\left(\sum_{n=0}^{\infty}p^n\left(h_n(x,t)\right)\right)_{xx}\right)\\+\left(\left(\sum_{n=0}^{\infty}p^n\left(h_n(x,t)\right)\right)_x\right)^2\end{array}\right]d\tau$$

On comparing the coefficient of various power of p, we get

$$p^0:h_0(x,t)=ax^2+bx+c$$

$$p^1:h_1(x,t)=\frac{K}{S_y}\left[-\int_0^t\left(h_0\left(\frac{\partial^2 h_0}{\partial x^2}\right)+\left(\frac{\partial h_0}{\partial x}\right)^2\right)d\tau\right]$$

$$h_1(x,t)=\frac{K}{S_y}\left(\left((b+2ax)^2+2a\left(ax^2+bx+c\right)\right)t\right)$$

$$p^2:h_2(x,t)=\frac{K}{S_y}\left[-\int_0^t\left(h_0\left(\frac{\partial^2 h_1}{\partial x^2}\right)+h_1\left(\frac{\partial^2 h_0}{\partial x^2}\right)+\left(2\frac{\partial h_0}{\partial x}\frac{\partial h_1}{\partial x}\right)\right)d\tau\right]$$

$$h_2(x,t)=\frac{K^2}{S_y^{\,2}}\left(\left(7b^2+8ac+36abx+36a^2x^2\right)at^2\right)$$

$$p^3:h_3(x,t)=\frac{K}{S_y}\left[-\int_0^t\left(\begin{array}{l}h_0\left(\frac{\partial^2 h_2}{\partial x^2}\right)+h_1\left(\frac{\partial^2 h_1}{\partial x^2}\right)\\+h_2\left(\frac{\partial^2 h_0}{\partial x^2}\right)+\left(2\frac{\partial h_0}{\partial x}\frac{\partial h_2}{\partial x}+\left(\frac{\partial h_1}{\partial x}\right)^2\right)\end{array}\right)d\tau\right]$$

$$h_3(x,t)=\frac{K^3}{3S_y^{\,3}}\left(\left(67b^2+56ac+324abx+324a^2x^2\right)2a^2t^3\right)$$

$$p^4:h_4(x,t)=\frac{K}{S_y}\left[-\int_0^t\left(\begin{array}{l}h_0\left(\frac{\partial^2 h_3}{\partial x^2}\right)+h_1\left(\frac{\partial^2 h_2}{\partial x^2}\right)\\+h_2\left(\frac{\partial^2 h_1}{\partial x^2}\right)+h_3\left(\frac{\partial^2 h_0}{\partial x^2}\right)\\+\left(2\frac{\partial h_0}{\partial x}\frac{\partial h_3}{\partial x}+2\frac{\partial h_1}{\partial x}\frac{\partial h_2}{\partial x}\right)\end{array}\right)d\tau\right]$$

$$h_4(x,t)=\frac{K^4}{3S_y^{\,4}}\left(52b^2+35ac+243abx+243a^2x^2\right)16a^3t$$

Continuing in similar approach we can obtain additional approximations. We get VHPM solution of equation (1) in the form of a series.

$$h(x,t) = h_0(x,t) + h_1(x,t) + h_2(x,t) + h_3(x,t) + h_4(x,t)$$

Considering the initial depth at $x = 0$ to be zero, we choose $c = 0$. Applying boundary conditions (2) and (3), we get $a = -1/6$ and $b = 1$. On substituting the value of a and b and assuming ratio $K/S_y = 1$ in above equation we get

$$h(x,t) = x - \frac{x^2}{6} + t\left(\left(1-\frac{x}{3}\right)^2 + \frac{1}{3}\left(-x+\frac{x^2}{6}\right)\right) - \frac{t^2}{6}\left(7-6x+x^2\right) + \frac{t^3}{54}\left(67-54x+9x^2\right)$$
$$-\frac{2t^4}{81}\left(52-\frac{81}{2}x+\frac{27}{4}x^2\right)$$

On adjusting the terms and rearranging them we can write the solution in the closed form as

$$h(x,t) = -\frac{1}{6}(x-3)^2(1+t)^{-1} + \frac{3}{2}(1+t)^{-1/3}$$

IV. Results and Discussion

In the current paper a mathematical model for stream–aquifer interaction by means of the Boussinesq equation is presented. A novel approach to the solution of the Boussinesq equation for a semi-infinite aquifer is found out by VHPM when a random function of time is taken as water head at the source and compared with the available exact solution. Table 1 disclose that the solution obtained by VHPM is in fine agreement with the available exact solution. The numerical values for the height of water table are observed for diverse distances and different time t as revealed in Table 2. The obtained results for height of water table obtained by VHPM for specific values of time and space are compared with the exact solution as shown in the table 1, it is observed that the solution obtained by VHPM is very close with the exact solution for l_x =3(m) and for t = 0.25day. Thus, the result obtained satisfies the boundary conditions and behave well with the physical phenomena for various values of time. The graphical representation of the effect of S_y and K are shown in figure (3) and figure (4) respectively. It is seen that with the increasing value of S_y the height of the water is increasing and with the increasing value of K the height of the water table is decreasing which is consistent with respect to the properties of the aquifer parameters. The effect of the ratio K/S_y is also observed and its graphical representation is shown is figure (5). From the graphical representation it is seen that the height of the water table in sandstone is maximum amongst the five materials considered i.e. the height of the water at x=3(m) is decreasing minimum from its initial value in comparison to other materials.

Table- I: Comparison of numerical values of water mound in an unconfined horizontal aquifer obtained by Exact solution and VHPM

Distance (x)(m)	$h(x,t)$ at **(t = 0.25 days)**	
	EXACT	**VHPM**
0	0.19247665008383383	0.19247665008383374
0.2	0.34714331675050053	0.3471433167505005
0.4	0.4911433167505006	0.49114331675050027
0.6	0.6244766500838338	0.6244766500838337
0.8	0.7471433167505006	0.7471433167505003
1	0.8591433167505006	0.8591433167505004
1.2	0.9604766500838338	0.9604766500838338
1.4	1.0511433167505002	1.0511433167505002
1.6	1.1311433167505007	1.1311433167505003
1.8	1.200476650083834	1.2004766500838338
2	1.2591433167505004	1.2591433167505004
2.2	1.3071433167505004	1.3071433167505004
2.4	1.3444766500838339	1.3444766500838337
2.6	1.3711433167505005	1.3711433167505003
2.8	1.3871433167505005	1.3871433167505003
3	1.3924766500838344	1.3924766500838337

Table- II: Numerical values of water head in an unconfined horizontal aquifer for different distance x at time $t = 0,\ldots,4.3$ days

	height $h(x,t)$											
x **(m)**	***t*=0 (days)**	***t*=0.4 (days)**	***t*=0.8 (days)**	***t*=1.2 (days)**	***t*=1.6 (days)**	***t*=1.8 (days)**	***t*=2 (days)**	***t*=2.8 (days)**	***t*=3.2 (days)**	***t*=4 (days)**	***t*=4.2 (days)**	***t*=4.3 (days)**
0	0	0.2694	0.3998	0.4715	0.4715	0.5285	0.5400	0.5665	0.5725	0.5772	0.5774	0.5773
0.5	0.458	0.5968	0.6544	0.6798	0.6623	0.6922	0.6928	0.6871	0.6817	0.6689	0.6655	0.6638
1	0.833	0.8646	0.8627	0.8503	0.8153	0.8261	0.8178	0.7858	0.7710	0.7439	0.7376	0.7345
1.5	1.125	1.073	1.0247	0.9829	0.9304	0.9303	0.9150	0.8625	0.8404	0.8022	0.7937	0.7896
2	1.333	1.2218	1.1405	1.0776	1.0076	1.0047	0.9844	0.9174	0.8900	0.8439	0.8338	0.8289
2.5	1.458	1.3111	1.2100	1.1344	1.0469	1.0494	1.0261	0.9503	0.9198	0.8689	0.8578	0.8525
3	1.5	1.3409	1.2331	1.1533	1.0483	1.0642	1.0400	0.9612	0.9297	0.8772	0.8658	0.8603

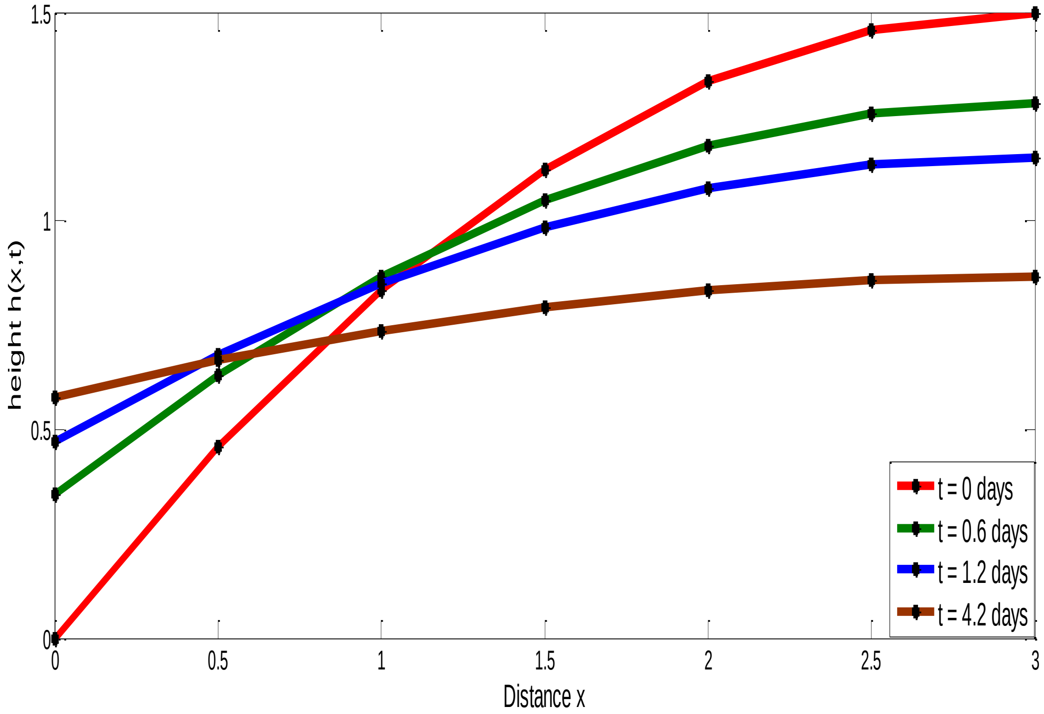

Fig. 2. Height of water table in an unconfined horizontal aquifer for different distance x

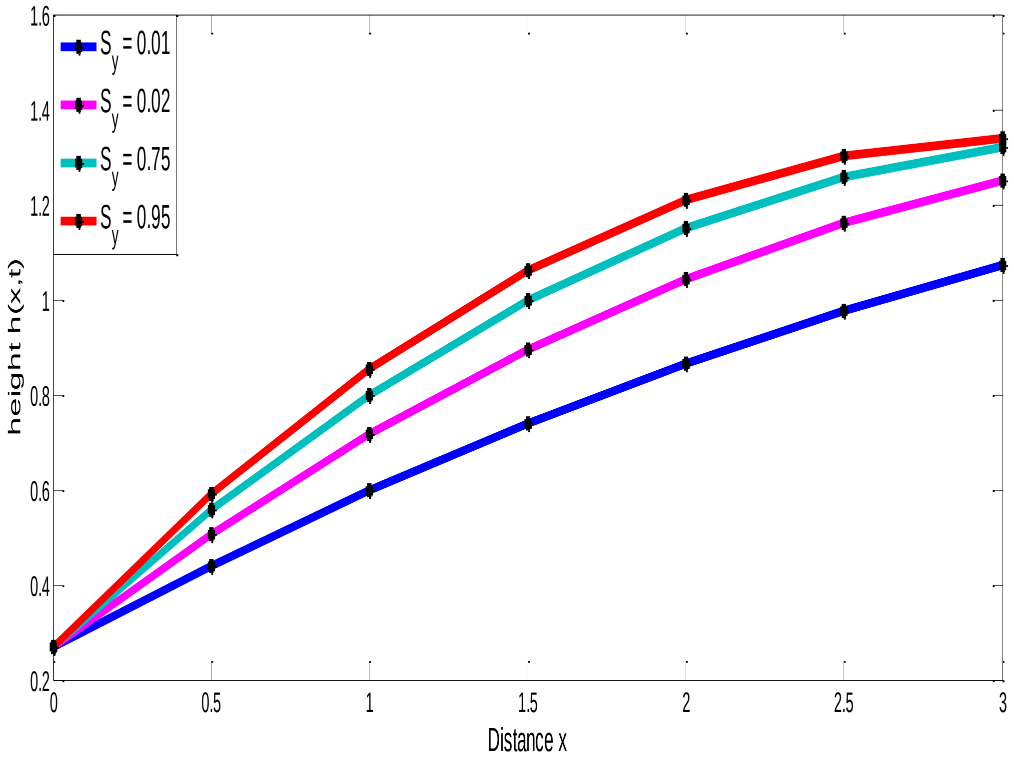

Fig. 3. Height of water table in an unconfined horizontal aquifer for different value of S_y

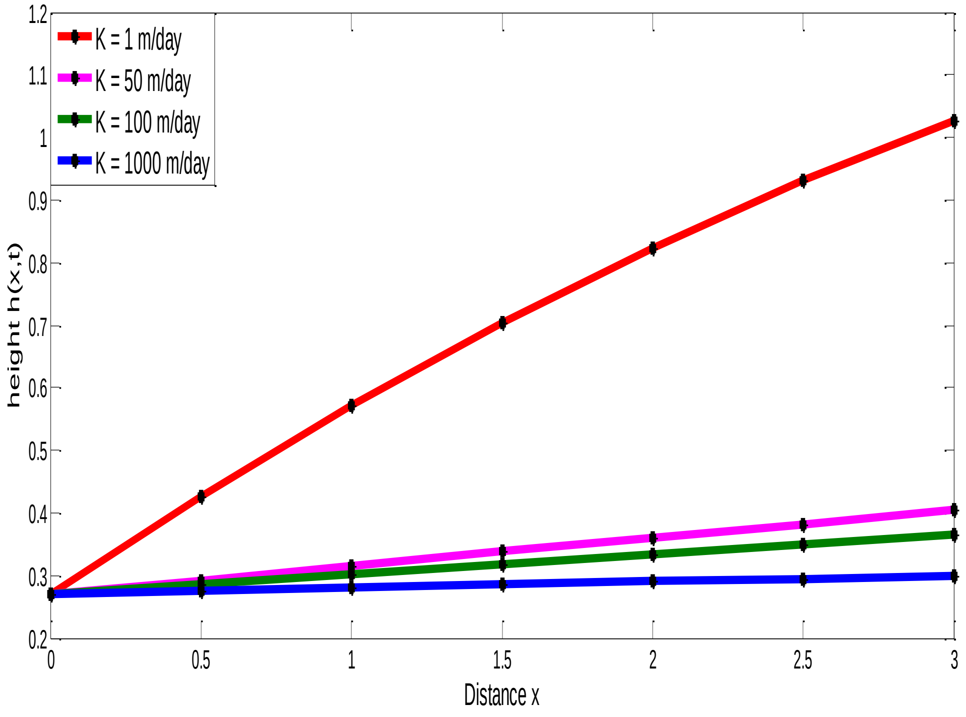

Fig. 4. Height of water table in an unconfined horizontal aquifer for different value of K

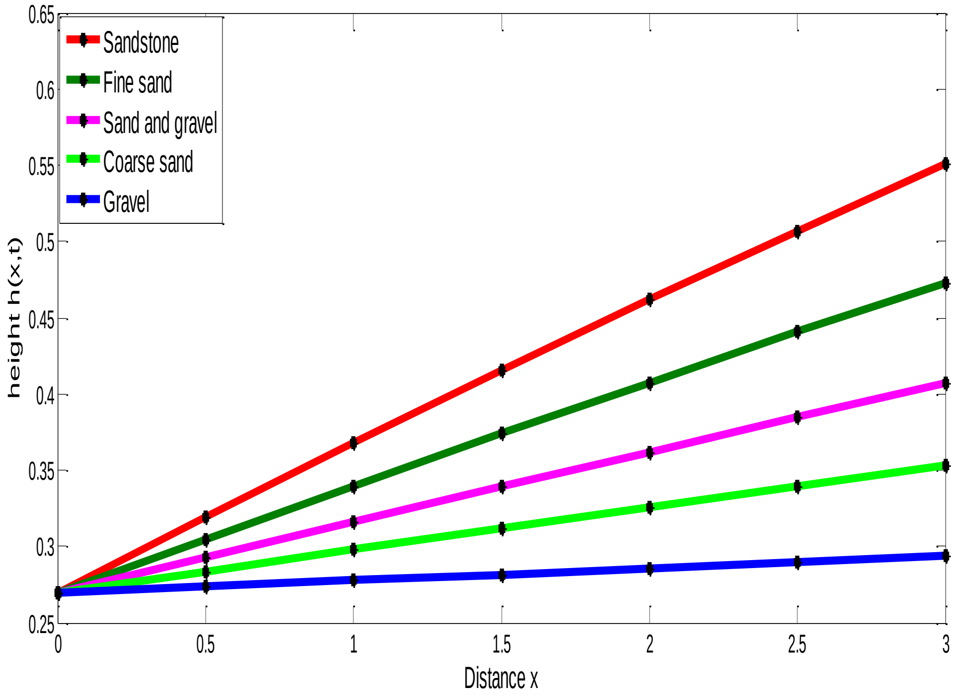

Fig.5. Height of water table in an unconfined horizontal aquifer for different values of value of ratio K/S_y

V. Conclusion

The approximate analytical solution of Boussinesq equation for horizontal aquifer is obtained by applying VHPM and compared with the available exact solution for the horizontal aquifer. It is concluded from the obtained result that VHPM is easy, accurate and convenient. From the results obtained it is concluded that the solution satisfies the boundary conditions and resembles well with the physical phenomena. The two important parameters viz. Hydraulic conductivity and Specific yield (S_y) are considered in the present groundwater flow problem. The approximate analytical solution is obtained by considering the ratio $\frac{K}{S_y} = 1$However, the sensitivity of these parameters is studied for five different ratios for five different samples. It is concluded that among the five samples considered the height of the water is maximum in sandstone for horizontal aquifer whereas it is maximum in gravel for sloping aquifer. Various authors have obtained the solution of Boussinesq with different boundary conditions. The present shown VHPM can be applied equally well with the suitable choice of initial condition.

The chief benefit of Variational Homotopy Perturbation Method is one has independence to choose ones initial approximation. We get mostly approximate analytical solution which converges rapidly. It should be noted that the method is competent of reducing the headache of the computational work as far as the typical methods are concerned with proper accuracy of the result. Thus VHPM can be applicable in different field of sciences.

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